7. *On Newman Algebras.* I

By F. M. SIOSON

Department of Mathematics, University of Florida

(Comm. by Kinjirô KUNUGI, M.J.A., Jan. 12, 1965)

1. Introduction. The axiomatics of Newman algebras has been the subject of a number of papers by M. H. Newman [3], [4], G. D. Birkhoff and G. Birkhoff [1], [2], Y. Wooyenaka [7], [8], and F. M. Sioson [6]. In the present communication, a Newman algebra will be considered as an algebraic system $(N, {}^{+}, {}^{\cdot}, {}^{-})$ with two binary operations $^{+}$ and $^{\cdot}$ and one unary operation $^{-}$.

For any postulate P of Newman algebras, let P^{+} (similarly $P^{\cdot}$) denote the proposition obtained from P by commuting all the additions (multiplications) occurring in it. Thus, for instance, if P is $x(y\bar{y}+x)=x$, then $P^{\cdot}$ is $(\bar{y}y+x)x=x$. Note that, in general, $P^{++}=P=P^{\cdot\cdot}$, $P^{+\cdot}=P^{\cdot+}$, and if no $^{+}$ (no $^{\cdot}$) occurs in P, then $P^{+}=P(P^{\cdot}=P)$. Obviously, the propositional transformations $^{+}$ and $^{\cdot}$ thus defined generate an abelian group G_4 with four elements.

The author has previously shown in [6] that the only independent systems of postulates or *equational bases* for Newman algebras one can choose out of the following nine equations

$$N_1\colon\ x(y+z)=xy+xz,$$

$$N_2\colon\ x(y+\bar{y})=x, \qquad \bar{N}_2\colon\ x+y\bar{y}=x,$$

$$N_3\colon\ xy=yx, \qquad \bar{N}_3\colon\ x+y=y+x,$$

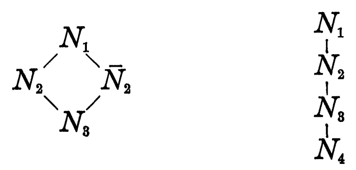
$$N_4\colon\ x(y\bar{y})=y\bar{y},$$

$$N_5\colon\ xx=x,$$

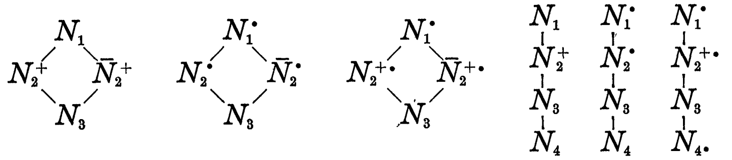
$$N_6\colon\ \bar{\bar{x}}=x,$$

$$\bar{N}_7\colon\ x+(y+z)=(x+y)+z,$$

are the systems:



and their transforms under the members of G_4:

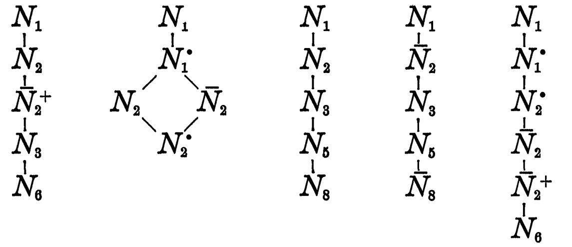


In fact, it can be shown that these are all the equational bases for Newman algebras with the least possible number of equations (i.e.

four) that can be chosen from the pool of nine equations above and their transforms under the group G_4. It was also shown in [6] that each of these eight Newman bases gives rise to a Boolean equational basis when one of the following equations, $x+x=x$, $x+yz=(x+y)(x+z)$, $x+(y+\bar{y})=y+\bar{y}$, $x+xy=x$, $x(x+y)=x$, is adjoined. Generally, of course, an axiomatization of Boolean algebras is always obtained after adjoining the equation $x+x=x$ to any independent system of axioms for Newman algebras, but they need not be independent.

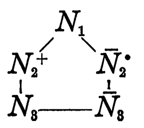
Y. Wooyenaka [7], [8] also gave two systems of generating equations for Newman algebras. Her axiom-system **I** is essentially an equational basis of Newman algebras, but, as we shall show later, one half of postulate C' (which is $\bar{N}_2^{\cdot}$ and $\bar{N}_2^{+\cdot}$ in our notation) is her axiom-system **II** is superfluous. The independence of her axiom system **II** has been achieved, moreover, by combining two equations in each of the postulates C' and C_1'. In our present considerations, we shall completely omit Wooyenaka's postulates F', D, D_1, and E'. Anyway, these postulates are implicit in the definition of any algebraic system. If one wants to worry about their independence in any of the equational bases we shall propose, then he can easily verify their independence by using the same models given by Wooyenaka for each of them. Furthermore, the independence-models we shall use all satisfy the postulates F', D_1, D, and E'.

To summarize, we shall demonstrate in this communication that by combining various transforms of the nine Newman equations given above, N_8: $x\bar{x}=y\bar{y}$, and $\bar{N}_8$: $x+\bar{x}=y+\bar{y}$, the following equational bases for Newman algebras arise



A **B** **C** **D** **E**

2. The Equational Basis A. To prove the equational completeness of system **A** for Newman algebras, it is easiest to derive our system **I** in [6], but due to the unavailability of this publication at present, we shall make our arguments dependent on Y. Wooyenaka's paper [7] and [8]. We shall show that Wooyenaka's equational basis



can be derived from **A**. To this end, it thus suffices to show that $\bar{N}_3$ follows from **A**.

2.1. $xx=x$.

$xx=x\bar{x}+xx=x(\bar{x}+x)=x(\bar{x}+\bar{\bar{x}})=x$ $(\bar{N}_2^+, N_1, N_6, N_2)$.

2.2. $x+\bar{x}=y+\bar{y}$.

$x+\bar{x}=(x+\bar{x})(y+\bar{y})=(y+\bar{y})(x+\bar{x})=y+\bar{y}$ (N_2, N_3, N_2).

2.3. $\overline{y\bar{y}}=y+\bar{y}$.

$\overline{y\bar{y}}=y\bar{y}+\overline{y\bar{y}}=y+\bar{y}$ $(\bar{N}_2^+, 2.2)$.

2.4. $x\bar{x}=y\bar{y}$.

$x\bar{x}=\overline{\overline{x\bar{x}}}=\overline{x+\bar{x}}=\overline{y+\bar{y}}=\overline{\overline{y\bar{y}}}=y\bar{y}$ $(N_6, 2.3, 2.2, 2.3, N_6)$.

2.5. $x+y\bar{y}=x$.

$x+y\bar{y}=xx+x\bar{x}=x(x+\bar{x})=x$ $(2.1\text{–}2.4, N_1, N_2)$.

2.6. $x(y\bar{y})=y\bar{y}$.

$x(y\bar{y})=x(x\bar{x})+x\bar{x}=x(x\bar{x}+\bar{x})=x\bar{x}=y\bar{y}$ $(2.4\text{–}2.5, N_1, \bar{N}_2^+, 2.4)$.

2.7. $(x+y)z=xz+yz$.

$(x+y)z=z(x+y)=zx+zy=xz+yz$ (N_3, N_1, N_3-N_3).

2.8. If $x+y=z\bar{z}$, then $x=y$.

(a) $x=x(y+\bar{y})=xy+x\bar{y}=xy+(x\bar{y}+y\bar{y})=xy+(x+y)\bar{y}=xy+(z\bar{z})\bar{y}$
$=xy+\bar{y}(z\bar{z})=xy+z\bar{z}=xy$ $(N_2, N_1, 2.5, 2.7,$ hypothesis, $N_3,$ $2.6, 2.5)$.

(b) $y=y(x+\bar{x})=yx+y\bar{x}=xy+(x\bar{x}+y\bar{x})=xy+(x+y)\bar{x}=xy+(z\bar{z})\bar{x}$
$=xy+\bar{x}(z\bar{z})=xy+z\bar{z}=xy$ $(N_2, N_1, N_3-\bar{N}_2^+, 2.7,$ hypothesis, $N_3, 2.6, 2.5)$.

Whence $x=y$.

2.8′. $x+y=(x+y)(y+x)$.

Since $z\bar{z}=(y+x)\overline{(y+x)}=y\overline{(y+x)}+x\overline{(y+x)}$ (2.4, 2.7), then $y\overline{(y+x)}=x\overline{(y+x)}$ by 2.8. This implies that $(x+y)\overline{(y+x)}=x\overline{(y+x)}+y\overline{(y+x)}=y\overline{(y+x)}+x\overline{(y+x)}=(y+x)\overline{(y+x)}$ (2.7, previous remarks, 2.7). Whence, $x+y=(x+y)((y+x)+\overline{(y+x)})=(x+y)(y+x)+(x+y)\overline{(y+x)}=(x+y)(y+x)+(y+x)\overline{(y+x)}=(x+y)(y+x)$ $(N_2, N_1,$ previous result, 2.5).

2.9. $x+y=y+x$.

$x+y=(x+y)(y+x)=(y+x)(x+y)=y+x$ $(2.8, N_3, 2.8')$.

AN_1. The Independence-Model of N_1 from N_2, $\bar{N}_2^+$, N_3, and N_6.

$+$	0	1
0	0	0
1	0	1

$\cdot$	0	1
0	0	1
1	1	0

y	$\bar{y}$
0	1
1	0

Here $1(1+0)\neq 11+10$.

AN_2. *The Independence-Model of* N_2 *from* N_1, $\bar{N}_2^+$, N_3, *and* N_6.

Consider the collection of all finite unions offinite, semi-infinite, and infinite open intervals on the real line, under the set theoretical operations of union (denoted by $^+$), of intersection (denoted by $\cdot$), and of the complement of the closure of a set (denoted by $^-$). Then $y\bar{y}=0$, and the equations N_1, $\bar{N}_2^+$, N_3, and N_6 are obvious. On the other hand,

$$(1, 3)\cdot((2, 4)+\overline{(2, 4)})\neq(1, 3).$$

AN_3. *The Independence-Model of N_3 from N_1, N_2, $\bar{N}_2^+$, and N_6.*

+	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	a	1
b	b	1	1	b

·	0	1	a	b
0	0	0	0	0
1	0	1	0	1
a	0	a	a	0
b	0	b	0	b

y	$\bar{y}$
0	1
1	0
a	b
b	a

Note that $1b\neq b1$ and $1a\neq a1$.

A$\bar{N}_2^+$. *The Independence-Model of $\bar{N}_2^+$ from N_1, N_2, N_3, and N_6.*

By interchanging the operations $+$ and $\cdot$ in the model AN_2, we then obtain

$$(1, 3)+(2, 4)\overline{(2, 4)}\neq(1, 3),$$

but the other equations are still verified easily.

AN_6. *The Independence-Model of N_6 from N_1, N_2, $\bar{N}_2^+$, and N_3.*

+	0	1
0	0	1
1	0	1

·	0	1
0	0	1
1	1	1

y	$\bar{y}$
0	0
1	0

For this model, note that $\bar{\bar{1}}\neq 1$.

It should be noted further that each of the models used in the independence proof of **A** satisfies the equation $x+x=x$. Since this last equation is independent of any formulation of Newman algebras, it follows that **A** together with the equation $x+x=x$ and all their duals and transforms under G_4 form equational bases for Boolean algebras.

(For the references, see the end of the next article.)