6. On Hahn-Banach Type Extension Theorem

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Recently, M. Antonovski, V. Boltjanski and T. Sarimsakov [1] introduced an important concept named *topological semifield*. The concept is quite useful for the discussion of functional analysis. In this short Note, we shall prove a Hahn-Banach type theorem.

Let X be a linear space on the real field, E a topological semifield. Suppose that a functional $f(x)$ on X takes on the value in E, then we can consider the homogeneous functional, i.e. $f(\alpha x)=\alpha f(x)$ for real α. As proved in ([1], p. 24), E contains the real field R^1 as a subsemifield, therefore $\alpha f(x)$ has the meaning in E. Hence additive and homogeneous functionals on a linear space are *well-defined*. Similarly, we can consider a functional $p(x)$ satisfying the following conditions:

(1) $$p(x+y)\ll p(x)+p(y),$$

(2) $$p(\alpha x)=\alpha p(x) \quad \text{for} \quad \alpha>0,$$

where the symbol $\ll$ means the order in E (see [1], p. 7).

We prove the following fundamental theorem which is similar with Hahn-Banach theorem.

Theorem. Let $p(x)$ be a functional on X satisfying the conditions (1) *and* (2), *$f(x)$ a linear functional defined on a linear subspace X_0 of X. If $f(x)\ll p(x)$ on X_0, then $f(x)$ has a linear extension $F(x)$ on X satisfying $F(x)\ll p(x)$ on X.*

Proof. Let $X-X_0\neq\phi$, and take an element $x_0\in X-X_0$, then each element x of the linear subspace $X_1=[X_0, x_0]$ generated by X_0 and x_0 is uniquely represented in the form of $x=\alpha x_0+x'(x'\in X_0)$. If $x', x''\in X_0$, then we have

$$\begin{aligned} f(x')+f(x'')=f(x'+x'')&\ll p((x_0+x')+(-x_0+x'')) \\ &\ll p(x_0+x')+p(-x_0+x''). \end{aligned}$$

Hence we have

$$f(x'')-p(-x_0+x'')\ll -f(x')+p(x_0+x').$$

Elements x', x'' are arbitrary in X_0 and then, by the result in ([1], p. 10). we obtain

$$m=\sup_{x''\in X_0}.(f(x'')-p(-x_0+x''))\ll \inf_{x'\in X_0}.(-f(x')+p(x_0+x'))=M,$$

where m, M are contained in E. Therefore, we can take an element ξ in E such that $m\ll\xi\ll M$. Define $f_1(x)$ as

$$f_1(x)=\alpha\xi+f(x')$$

for $x=\alpha x_0+x'(x'\in x_0)$. The functional $f_1(x)$ is clearly additive and homogeneous on X_1. Hence f_1 is a linear extension of f. Let $\alpha>0$, then, by properties of topological semifield,

$$
\begin{aligned}
f_1(x)&=\alpha\xi+f(x')\ll\alpha M+f(x')\\
&\ll\alpha\left[-f\left(\frac{x'}{\alpha}\right)+p\left(x_0+\frac{x'}{\alpha}\right)\right]+f(x')\\
&=-f(x')+p(\alpha x_0+x')+f(x')=p(\alpha x_0+x').
\end{aligned}
$$

This shows $f_1(x)\ll p(x)$ for $\alpha>0$. Similarly we have $f_1(x)\ll p(x)$ for $\alpha<0$. Hence $f_1(x)\ll p(x)$ on X_1. By Zorn's lemma, we can extend f on the whole X. We complete the proof.

Reference

[1] М. Я. Антоновский, В. Г. Болтянский, Т. А. Сарымсаков: Топологические полуполя. СамГУ. Ташкент (1960).