5. The Plancherel Formula for the Universal Covering Group of De Sitter Group

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(Comm. by Kinjirô KUNUGI, M.J.A., Jan. 12, 1965)

In his recent paper [3], R. Takahashi conjectured the explicit Plancherel formula for the universal covering group of De Sitter group.

The purpose of this paper is to prove that this formula is actually the Plancherel formula of the group.

The method in the present paper can be applied for other groups. For simplicity, however, we confine our considerations only to the above mentioned group.

1. Let G be the universal covering group of De Sitter group realized in [3].

We define three one-parameter subgroups whose generic elements are;

$$m_\varphi=\begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{i\varphi/2}\end{pmatrix}, \qquad a_t=\begin{pmatrix} \mathrm{ch}t/2 & \mathrm{sh}t/2 \\ \mathrm{sh}t/2 & \mathrm{ch}t/2\end{pmatrix}, \qquad u_\theta=\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2}\end{pmatrix}$$

respectively and denote by H_0, H_1, H_2 the left invariant infinitesimal transformations defined by these subgroups. Put

$$A_1=\{a_t m_\varphi;\ t, \varphi\in \boldsymbol{R}\}, \qquad A_2=\{u_\theta m_\varphi;\ \theta, \varphi\in \boldsymbol{R}\}.$$

Then A_1 and A_2 are the non conjugate Cartan subgroups of G (see [1 (b)]). Every Cartan subgroup of G is conjugate with either A_1 or A_2 (see [2]). We put $G_k=\bigcup_{g\in G} gA_kg^{-1}$ $(k=1, 2)$.

Let $U_{n,\nu}$ and $T_{n,p}$ be the characters of the representations $U^{n,3/2+i\nu}$ and $T^{n,0,p}\oplus T^{0,n,p}$ defined in [3] respectively, then there are locally summable functions $\chi^{(1)}_{n,\nu}, \chi^{(2)}_{n,p}$ on G such that

$$U_{n,\nu}(f)=\int_G f(g)\chi^{(1)}_{n,\nu}(g)dg, \qquad T_{n,p}(f)=\int_G f(g)\chi^{(2)}_{n,p}(g)dg,$$

where dg is the Haar measure on G (cf. [1(f)]). Let $\mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2$ be the Lie algebras of G, A_1, A_2 respectively.

There exists a Cartan involution θ of $\mathfrak{g}$ such that $\theta\mathfrak{h}_k=\mathfrak{h}_k$ $(k=1, 2)$. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$.

We can select compatible orderings in the dual spaces of $\mathfrak{h}_k\cap\mathfrak{p}$ and $\mathfrak{h}_k\cap\mathfrak{p}+i\mathfrak{h}_k\cap\mathfrak{k}$ (see [1(d)]). Let P_k be all positive roots in this order. Put $P_k^+=\{\alpha\in P_k;\ \alpha(\mathfrak{h}_k\cap\mathfrak{p})\neq\{0\}\}$, then $P_k-P_k^+$ is the disjoint sum of the set P_k^0 of all non compact positive roots and the set P_k^- of all compact positive roots (see 1(b)). We put

$$\Delta_k(\exp H)=\left|\prod_{\alpha\in P_k^+}(e^{\alpha(H)/2}-e^{-\alpha(H)/2})\right|\prod_{\alpha\in P_k^0\cup P_k^-}(e^{\alpha(H)/2}-e^{-\alpha(H)/2})$$

Due to T. Hirai, the values of the characters on regular elements in Cartan subgroups are as follows;

$$\chi^{(1)}_{n,\nu}(a_t m_\varphi)=\frac{i}{\Delta_1(a_t m_\varphi)}\left\{\cos \nu t \sin\left(n+\frac{1}{2}\right)\varphi\right\},$$

$$\chi^{(1)}_{n,\nu}(u_\theta m_\varphi)=0,$$

$$\chi^{(2)}_{n,p}(a_t m_\varphi)=\frac{i}{\Delta_1(a_t m_\varphi)}\left\{e^{-(p-\frac{1}{2})|t|}\sin\left(n+\frac{1}{2}\right)\varphi-e^{-(n+\frac{1}{2})|t|}\sin\left(p-\frac{1}{2}\right)\varphi\right\},$$

$$\chi^{(2)}_{n,p}(u_\theta m_\varphi)=\frac{-1}{\Delta_2(u_\theta m_\varphi)}\left\{\sin\left(p-\frac{1}{2}\right)\theta\sin\left(n+\frac{1}{2}\right)\varphi - \sin\left(n+\frac{1}{2}\right)\theta\sin\left(p-\frac{1}{2}\right)\varphi\right\}.$$

From these, we can deduce the following formulas (A) (see [1 (a)]);

$$U_{n,\nu}(f)=-i\int_0^\infty\int_0^{2\pi}\cos \nu t\sin\left(n+\frac{1}{2}\right)\varphi F_f^{(1)}(a_t m_\varphi)dtd\varphi,$$

$$\text{(A)}\quad T_{n,p}(f)=-i\int_0^\infty\int_0^{2\pi}\left\{e^{-(p-\frac{1}{2})t}\sin\left(n+\frac{1}{2}\right)\varphi - e^{-(n+\frac{1}{2})t}\sin\left(p-\frac{1}{2}\right)\varphi\right\}F_f^{(1)}(a_t m_\varphi)dtd\varphi$$
$$-\frac{1}{4}\int_{-2\pi}^{2\pi}\int_{-2\pi}^{2\pi}\left\{\sin\left(p-\frac{1}{2}\right)\theta\sin\left(n+\frac{1}{2}\right)\varphi - \sin\left(n+\frac{1}{2}\right)\theta\sin\left(p-\frac{1}{2}\right)\varphi\right\}F_f^{(2)}(u_\theta m_\varphi)d\theta d\varphi,$$

for all $f\in C_c^\infty(G)$.

In this formula,

$$F_f^{(k)}(h)=\Delta_k(h)\int_{G/A_k} f(h^{x^*})d\mu_k(x^*)\quad \text{for } h\in A_k',\quad (k=1,2)$$

where x^* is an element of G/A_k and $h^{x^*}=xhx^{-1}$ if $x^*=xA_k$, and μ_k is the invariant measure on G/A_k.

2. Let A_2'' be the set of all points $h=\exp H\in A_2$ such that $\prod_{\alpha\in P_2^0}(e^{\alpha(H)/2}-e^{-\alpha(H)/2})\neq 0$, then $u_\theta m_\varphi\in A_2''$ if and only if $\theta\not\equiv 0$, $\varphi\not\equiv 0$ (mod 2π).

Let B be the Killing form on $\mathfrak{g}^C$. For each $\alpha\in P_k$, there exists a unique element $H_\alpha\in\mathfrak{h}_k^C$ such that $B(H,H_\alpha)=\alpha(H)$ for all $H\in\mathfrak{h}_k^C$.

Put $\partial(\pi_k)=\prod_{\alpha\in P_k}H_\alpha$, where the products are meant to be those of left invariant differential operators.

When D is a differential operator on A_k, put $f(h,D)=(Df)(h)$. From the general theory due to Harish Chandra (see [1 (d) (e) (f)]), we have the following lemma.

Lemma 1. (1) $F_f^{(1)}$ can be extended to a function of class C^∞ on A_1 with the compact support.

(2) Let B be any connected component of A_2', then $F_f^{(2)}$ and its derivatives of arbitrary degree can be extended to continuous functions on the closure of B in A_2 with the compact supports, which are class C^∞ on A_2''.

(3) There exists a real number $c \neq 0$ such that

$$\lim_{h\to e} F_f^{(2)}(h, \partial(\pi_2)) = cf(e). \qquad (h \in A_2')$$

The following lemma plays an essential role in the present paper.

Lemma 2. Let P be any polynomial in two indeterminates, then the series

$$\sum_{n\geq p>0} P(n, p) T_{n,p}(f) \qquad (f \in C_c^\infty(G))$$

is absolutely convergent.

Making use of lemma 1 and lemma 2, we get the following.

Theorem

(1) $$2\nu\left(n+\frac{1}{2}\right)\left[\left(n+\frac{1}{2}\right)^2+\nu^2\right]U_{n,\nu}(f)$$
$$=6^4\int_0^\infty\int_{-2\pi}^{2\pi} \sin \nu t \cos\left(n+\frac{1}{2}\right)\varphi F_f^{(1)}(a_t m_\varphi, \partial(\pi_1))dtd\varphi,$$

(2) $$4\,|\,lm(l^2-m^2)\,|\,T_{n,p}(f)$$
$$=-2\times 6^4\int_0^\infty\int_{-2\pi}^{2\pi}\{e^{-|l|t}\cos m\varphi + e^{-|m|t}\cos l\varphi\}F_f^{(1)}(a_t m_\varphi, \partial(\pi_1))dtd\varphi$$
$$+6^4\int_{-2\pi}^{2\pi}\int_{-2\pi}^{2\pi}\{\cos l\theta \cos m\varphi + \cos m\theta \cos l\varphi\}F_f^{(2)}(u_\theta m_\varphi, \partial(\pi_2))d\theta d\varphi,$$

where l, m are half integers such that $l-m \in \boldsymbol{Z}$, and $\mathrm{Max}\{|\,l\,|, |\,m\,|\} = n+\frac{1}{2}$, $\mathrm{Min}\{|\,l\,|, |\,m\,|\} = p-\frac{1}{2}$.

Lemma 3. $F_f^{(1)}(a_{-t}m_\varphi) = F_f^{(1)}(a_t m_\varphi)$, $F_f^{(1)}(a_t m_{-\varphi}) = -F^{(1)}(a_t m_\varphi)$,
$F_f^{(2)}(u_{-\theta}m_{-\varphi}) = F_f^{(2)}(u_\theta m_\varphi)$, $F_f^{(2)}(u_\varphi m_\theta) = -F_f^{(2)}(u_\theta m_\varphi)$.

From lemma 1 and lemma 3, we can deduce the following lemma.

Lemma 4.

Let r and s be non negative integers, then;

(1) $\lim_{t\to 0} F_f^{(1)}(a_t m_\varphi, H_0^r H_1^{2s+1}) = 0$,

(2) $\lim_{\varphi\to\beta} F_f^{(1)}(a_t m_\varphi, H_0^{2r} H_1^s) = 0$ if $\beta = 0$ or $\pm 2\pi$.

Making use of (1) in lemma 1, we can easily prove (1) in Theorem by integration by parts from (A).

However $F_f^{(2)}$ is, in general, not of class C^∞ and we can not expect even its continuity. The idea of the proof of (2) in Theorem can be explained as follows. Let l and m be the half integers such that $l-m \in \boldsymbol{Z}$ and put

$$\mathrm{Max}\{|\,l\,|, |\,m\,|\} = n+\frac{1}{2}, \qquad \mathrm{Min}\{|\,l\,|, |\,m\,|\} = p-\frac{1}{2}.$$

From the second formula in (A), using (2) in lemma 1, lemma 3 and 4 we can deduce the following (B) by integration by parts.

(B) $4\,|\,lm(l^2-m^2)\,|\,T_{n,p}(f)$

$$=2i\int_0^\infty\int_{-2\pi}^{2\pi}\{e^{-|l|t}\cos m\varphi+e^{-|m|t}\cos l\varphi\}F_f^{(1)}(a_t m_\varphi,\ H_0H_1(H_0^2+H_1^2))dtd\varphi$$
$$+\int_{-2\pi}^{2\pi}\int_{-2\pi}^{2\pi}\{\cos l\theta\cos m\varphi+\cos m\theta\cos l\varphi\}F_f^{(2)}(u_\theta m_\varphi,\ H_0H_2(H_0^2-H_1^2))d\theta d\varphi$$
$$+2\int_{-2\pi}^{2\pi}(\cos l\varphi+\cos m\varphi)\{J_f^m(m_\varphi, H_0(H_0^2+H_2^2))+iK_f(m_\varphi, H_0(H_0^2+H_2^2))\}d\varphi$$
$$+2\int_{-2\pi}^{2\pi}(l^2\cos m\varphi+m^2\cos l\varphi)\{J_f^m(m_\varphi,\ H_0)+iK_f(m_\varphi,\ H_0)\}d\varphi+C_f^m$$

where,

$$J_f^m(m_\varphi, D)=\lim_{\varepsilon\downarrow 0}\{F_f^{(2)}(u_\varepsilon m_\varphi, D)-F_f^{(2)}(u_{-\varepsilon}m_\varphi, D)$$
$$-(-1)^{2m}[F_f^{(2)}(u_{2\pi-\varepsilon}m_\varphi, D)-F_f^{(2)}(u_{-2\pi+\varepsilon}m_\varphi, D)]\},$$
$$K_f(m_\varphi, D)=\lim_{\varepsilon\to 0}F_f^{(1)}(a_\varepsilon m_\varphi, D),$$

for any differential operator D on A_2 and C_f^m is the constant with the following property,

$$C_f^r=C_f^s$$

for all half integers r, s such as $r-s\in \boldsymbol{Z}$. We compare the order of each term in (B) when l intends to infinity under certain conditions. Then, using lemma 1 and Theorem 1, we can show from Riemann-Lebesque theorem that last three terms on the right hand side in (B) must be all zero.

Since $H_0H_1(H_0^2+H_1^2)=i6^4\partial(\pi_1)$, $H_0H_2(H_0^2-H_2^2)=6^4\partial(\pi_2)$, we get (2) in Theorem.

3. Now we shall give a brief outline of the main steps in the determination of the Plancherel measure for G.

If we sum up each term in (2) of Theorem with respect to l, m, then we have the following formulas (1), (2) and (3).

(1) $$\sum_{2l,2m,l-m\in\boldsymbol{Z}}|\,lm(l^2-m^2)\,|\,T_{n,p}(f)$$
$$=8\sum_{n\ge p\ge 1}\left(n+\frac{1}{2}\right)\left(p-\frac{1}{2}\right)(n+p)(n-p+1)T_{n,p}(f)$$

(2) $$\sum_{2l,2m,l-m\in\boldsymbol{Z}}\int_0^\infty\int_{-2\pi}^{2\pi}\{e^{-|l|t}\cos m\varphi+e^{-|m|t}\cos l\varphi\}F_f^{(1)}(a_t m_\varphi,\ \partial(\pi_1))dtd\varphi$$
$$=4\sum_{\substack{m\ge 1\\ 2m\equiv 0 \pmod 2}}\lim_{k\to\infty}\int_0^\infty\int_{-2\pi}^{2\pi}\frac{e^{t/2}+e^{t/2}-2e^{-(k+\frac{1}{2})t}}{e^{t/2}-e^{-t/2}}\cos m\varphi F_f^{(1)}(a_t m_\varphi,\ \partial(\pi_1))dtd\varphi$$
$$+8\sum_{\substack{m>0\\ 2m\equiv 1 \pmod 2}}\lim_{k\to\infty}\int_0^\infty\int_{-2\pi}^{2\pi}\frac{1-e^{-kt}}{e^{t/2}-e^{-t/2}}\cos m\varphi F_f^{(1)}(a_t m_\varphi,\ \partial(\pi_1))dtd\varphi$$
$$=4\int_0^\infty\coth t/2\left\{\sum_{\substack{n>0\\ 2n\equiv 1 \pmod 2}}\int_{-2\pi}^{2\pi}\cos\left(n+\frac{1}{2}\right)\varphi F_f^{(1)}(a_t m_\varphi,\ \partial(\pi_1))\,d\varphi\right\}dt$$
$$+4\int_0^\infty\operatorname{cosech} t/2\left\{\sum_{\substack{n\ge 0\\ 2n\equiv 0 \pmod 2}}\int_{-2\pi}^{2\pi}\cos\left(n+\frac{1}{2}\right)\varphi F_f^{(1)}(a_t m_\varphi,\ \partial(\pi_1))d\varphi\right\}dt.$$

In the above deduction, we made use of lemma 1 and the fact that

$$\int_{-2\pi}^{2\pi} F_f^{(1)}(a_t m_\varphi, \partial(\pi_1))d\varphi = 0 \qquad \text{and the following.}$$

Lemma 5. $\frac{1}{t}F_f^{(1)}(a_t m_\varphi, \partial(\pi_1))$ is extended to a continuous function with the compact support on A_1.

This lemma is an immediate consequence of lemma 1 and lemma 4. In the following, lemma 1 is used, over again.

$$\begin{aligned}
&\sum_{2l,2m,l-m\in Z}\int_{-2\pi}^{2\pi}\int_{-2\pi}^{2\pi}\{\cos l\theta\cos m\varphi+\cos m\theta\cos l\varphi\}F_f^{(2)}(u_\theta m_\varphi, \partial(\pi_2)d\theta d\varphi\\
&=8\sum_{2l\equiv 2m\equiv 1(\mathrm{mod}2)}\int_0^{2\pi}\int_0^{2\pi}\cos l\theta\cos m\varphi F_{f_+}^{(2)}(u_\theta m_\varphi, \partial(\pi_2))d\theta d\varphi\\
&\quad+8\sum_{2l\equiv 2m\equiv 0(\mathrm{mod}2)}\int_0^{2\pi}\int_0^{2\pi}\cos l\theta\cos m\varphi F_{f_-}^{(2)}(u_\theta m_\varphi, \partial(\pi_2))d\theta d\varphi\\
&=32\pi^2\Big\{\lim_{(\theta,\varphi)\to(0,0)}F_{f_+}^{(2)}(u_\theta m_\varphi, \partial(\pi_2))+\lim_{(\theta,\varphi)\to(0,0)}F_{f_-}^{(2)}(u_\theta m_\varphi, \partial(\pi_2))\Big\}\quad(u_\theta m_\varphi\in A_2'),\\
&=32\pi^2 c\{f_+(e)+f_-(e)\}=32\pi^2 cf(e),
\end{aligned}$$

where

$$f_+(g)=\frac{1}{2}f(g)+f(\gamma g)$$

$$f_-(g)=\frac{1}{2}f(g)-f(\gamma g)\qquad\text{for } \gamma=\begin{pmatrix}-0 & 0\\ 0 & -1\end{pmatrix}.$$

From (1), (2), (3) and Theorem, we have;

$$\begin{aligned}
(4)\quad 4\pi^2\times 6^4 cf(e)&=4\sum_{n\ge p\ge 1}\Big(n+\frac{1}{2}\Big)\Big(p-\frac{1}{2}\Big)(n+p)(n-p+1)T_{n,p}(f)\\
&+\int_0^\infty \coth t/2\Big\{\sum_{\substack{n>0\\2n\equiv 1(\mathrm{mod}2)}}6^4\int_{-2\pi}^{2\pi}\cos\Big(n+\frac{1}{2}\Big)\varphi F_f^{(1)}(a_t m_\varphi, \partial(\pi_1))d\varphi\Big\}dt\\
&+\int_0^\infty \mathrm{cosech}\, t/2\Big\{\sum_{\substack{n\ge 0\\2n\equiv 0(\mathrm{mod}2)}}6^4\int_{-2\pi}^{2\pi}\cos\Big(n+\frac{1}{2}\Big)\varphi F_f^{(1)}(a_t m_\varphi, \partial(\pi_1))d\varphi\Big\}dt.
\end{aligned}$$

From (1) in Theorem, we get

$$\begin{aligned}
(5)\quad &2\sum_{\substack{n>0\\2n\equiv 1(\mathrm{mod}2)}}\nu\Big(n+\frac{1}{2}\Big)\Big[\Big(n+\frac{1}{2}\Big)^2+\nu^2\Big]U_{n,\nu}(f)\\
&\quad=\int_0^\infty \sin\nu t\Big\{\sum_{\substack{n>0\\2n\equiv 1(\mathrm{mod}2)}}6^4\int_{-2\pi}^{2\pi}\cos\Big(n+\frac{1}{2}\Big)\varphi F_f^{(1)}(a_t m_\varphi, \partial(\pi_1))d\varphi\Big\}dt,\\
&2\sum_{\substack{n\ge 0\\2n\equiv 0(\mathrm{mod}2)}}\nu\Big(n+\frac{1}{2}\Big)\Big[\Big(n+\frac{1}{2}\Big)^2+\nu^2\Big]U_{n,\nu}(f)\\
&\quad=\int_0^\infty \sin\nu t\Big\{\sum_{\substack{n\ge 0\\2n\equiv 0(\mathrm{mod}2)}}6^4\int_{-2\pi}^{2\pi}\cos\Big(n+\frac{1}{2}\Big)\varphi F_f^{(1)}(a_t m_\varphi, \partial(\pi_1))d\varphi\Big\}dt.
\end{aligned}$$

From the classical theory of Fourier transform, using lemma 5, we can finally derive the following formula from (4), (5).

$$4\times 6^4\pi^2 cf(e)=2\sum_{n\geq 0}(2n+1)\int_0^\infty U_{n,\nu}(f)\left[\left(n+\frac{1}{2}\right)^2+\nu^2\right]\nu\operatorname{th}(\pi(\nu+ni))d\nu$$
$$+\sum_{n\geq 1}(2n+1)\sum_{n\geq p\geq 1}(2p-1)(n+p)(n-p+1)T_{n,p}(f).$$

Let $d^{n,o,p}$ be the formal degree of $T^{n,o,p}$ (see [1(c)]). From Remark 5.2 in [3] (p. 431), we have

$$d^{n,o,p}=(2n+1)(2p-1)(n+p)(n-p+1)/16\pi^2,$$

under the normalization of the Haar measure of G that is introduced in [3]. Hence, if we fix such normalization, we have $c=\frac{4}{6^4}$ from the uniqueness of the Plancherel measure.

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