# 36. On Closures of Vector Subspaces. II

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5. We shall prove in this section the following theorem.[1)]

THEOREM 6. *Let $M$ be an infinite dimensional vector subspace of a vector space $E$, and let $\tau_0$ be a locally convex Hausdorff topology on $M$. Let us denote by $M'$ the dual of $M$ for the topology $\tau_0$, and by* $\operatorname{codim}(M')$ *the codimension of $M'$ in $M^*$.*

1° *If* $\operatorname{codim}(M)$ *is infinite, then* $\operatorname{codim}(M)\leqq 2^{\operatorname{codim}(M')}$ *implies that for every projection $p$ of $E$ onto $M$, there exists a locally convex Hausdorff topology $\tau$ on $E$ such that $M$ is dense in $E$ for the topology $\tau$ and $p$ is continuous for the topologies $\tau$ and $\tau_0$.*

*If* $\operatorname{codim}(M)$ *is finite, then* $\operatorname{codim}(M)\leqq\operatorname{codim}(M')$ *implies the same conclusion.*

*Conversely*

2° *If there exists a locally convex Hausdorff topology $\tau$ on $E$ such that $M$ is dense in $E$ for the topology $\tau$ and a projection $p$ of $E$ onto $M$ is continuous for the topologies $\tau$ and $\tau_0$, then either* $\operatorname{codim}(M)\leqq 2^{\operatorname{codim}(M')}$ *or* $\operatorname{codim}(M)\leqq\operatorname{codim}(M')$ *according as* $\operatorname{codim}(M)$ *is infinite or finite.*

Proof of 1°. Suppose first that the dimension of the vector subspace $N=p^{-1}(0)$ is infinite. The inequality $\dim(N)\leqq 2^{\operatorname{codim}(M')}$ shows that there exists a vector subspace $N'$ of $N^*$ such that $\dim(N')\leqq\operatorname{codim}(M')$ and the dual system $(N, N')$ is separated.[2)] Let $B_{N'}$ be a base of $N'$; then, since $\dim(N')\leqq\operatorname{codim}(M')$, we can find a linearly independent subset $B$ of an algebraic supplement of $M'$ in $M^*$ with cardinal number $\dim(N')$. Let $\varphi$ be a one-to-one mapping of $B_{N'}$ onto $B$. We define, for each $y'\in B_{N'}$, a linear functional $\bar{y}'$ on $E$ by setting

$$\langle x, \bar{y}'\rangle=\begin{cases}\langle x, \varphi(y')\rangle & \text{for } x\in M,\\ \langle x, y'\rangle & \text{for } x\in N.\end{cases}$$

1) This is a generalization of Theorem 1 of S. Kasahara: Locally convex metrizable topologies which make a given vector subspace dense. Proc. Japan Acad., **40**, 718-722 (1964); to this paper, corrections should be made as follows: Page 718, 'arized' should read 'arisen', and page 719, 'powder' should read 'power'.

2) See Lemma 4 of S. Kasahara: On closures of vector subspaces, I. Proc. Japan Acad., **40**, 723-727 (1964); the preceding sentence of Lemma 4 which begins with the word 'Consequently' should read as follows: Consequently, if the dual system $(E, E')$ is separated, we have $\dim(E)\leqq\cdots$.

Then the weakest topology $\tau$ on $E$ which makes the mapping $p$ and linear functionals $\bar{y}'(y' \in B_{N'})$ continuous possesses the required property. To see this, it will suffice to prove that $\tau$ is a Hausdorff topology which makes $M$ dense in $E$. It is easy to see that the mapping $x' \to x' \circ p$ of $M'$ into $E^*$ is continuous for the weak topologies $\sigma(M', M)$ and $\sigma(E^*, E)$. Therefore, if $A'$ is a $\sigma(M', M)$-compact subset of $M'$, then $A' \circ p = \{x' \circ p;\, x' \in A'\}$ is a $\sigma(E^*, E)$-compact subset of $E^*$. Consequently, for every closed convex and circled neighborhood $U$ of $0 \in M$ for the topology $\tau_0$, we have

$$(p^{-1}(U))^\circ = (p^{-1}(U^{\circ\circ}))^\circ = (U^\circ \circ p)^{\circ\circ} = U^\circ \circ p.$$

It follows that the dual $E'$ of $E$ for the topology $\tau$ is the vector subspace of $E^*$ spanned by the set $\{x' \circ p;\, x' \in M'\} \smile \{\bar{y}';\, y' \in B_{N'}\}$. Now to prove that the topology $\tau$ is Hausdorff, it will be sufficient to show that there exists, for each non-zero element $x$ of $N$, an element $x' \in E'$ such that $\langle x, x' \rangle \neq 0$. But this is an immediate consequence of the separatedness of the dual system $(N, N')$: in fact, we can find an element $y' \in B_{N'}$ for which we have $0 \neq \langle x, y' \rangle = \langle x, \bar{y}' \rangle$. It remains only to prove that the vector subspace $M$ is dense in $E$ for the topology $\tau$. Let $x'_0$ be an element of $E'$ which vanishes on $M$. Then we can find $x' \in M'$ and $y'_1, \cdots, y'_n \in B_{N'}$ such that $x'_0 = x' \circ p + \sum_{i=1}^n \lambda_i \bar{y}'_i$, and hence we have, for every $x \in M$,

$$0 = \langle x, x'_0 \rangle = \langle x, x' \circ p + \sum_{i=1}^n \lambda_i \bar{y}'_i \rangle = \langle x, x' + \sum_{i=1}^n \lambda_i \varphi(y'_i) \rangle.$$

In other words, the linear functional $x' + \sum_{i=1}^n \lambda_i \varphi(y'_i)$ on $M$ is the zero element of $M^*$, and so we have $x' = 0$ and $\lambda_1 = \cdots = \lambda_n = 0$, since the set $\{x', \varphi(y'_1), \cdots, \varphi(y'_n)\}$ is linearly independent. Consequently we have $M^\circ \frown E' = \{0\}$, which shows that $M$ is dense in $E$ for the topology $\tau$.

Suppose now that the dimension of the vector subspace $N$ is finite. Then we have $\dim(N^*) = \dim(N) \leqq \operatorname{codim}(M')$, and hence it suffices to take $N' = N^*$ in the proof of the case where $\dim(N)$ is infinite.

Proof of 2°. Suppose that the dimension of the vector subspace $N = p^{-1}(0)$ is infinite. Let $E'$ be the dual of $E$ for the topology $\tau$, and let $x' \in E'$, $A' \subseteq E'$. We denote by $x'|_M$ the restriction of $x'$ to $M$, and by $A'|_M$ the set of all restrictions $x'|_M$ of $x' \in A'$ to $M$.

Let $N'$ be an algebraic supplement of $N^\circ$ in $E'$. We shall show that $M' \frown (N'|_M) = \{0\}$. Let $x' \in M' \frown (N'|_M)$. Then since $x' \in M'$, we can write $x' = (x' \circ p)|_M$. On the other hand, since $x' \in N'|_M$, we have $x' = x'_1|_M$ for some $x'_1 \in N'$. Hence we have $x' \circ p = x'_1$, because the vector subspace $M$ is dense in $E$. But then, since $x' \circ p \in N^\circ$ and $x'_1 \in N'$, it follows that $x' \circ p = 0$, and so $x' = 0$. Thus $M' \frown (N'|_M) =$

$\{0\}$. Consequently we have

$$\operatorname{codim}(M') \geqq \dim(N'|_M). \tag{1}$$

Now it is clear that the mapping $y' \to y'|_M$ of $N'$ onto $N'|_M$ is linear. Moreover, this mapping is one-to-one, since $M$ is dense in $E$. Therefore we have

$$\dim(N'|_M) = \dim(N'). \tag{2}$$

Since the dual system $(E, E')$ is separated, for every non-zero element $x$ of $N$, we can find an $x' \in E'$ such that $\langle x, x' \rangle \neq 0$; but then we can write $x' = z' + y'$, where $z' \in N^\circ$ and $y' \in N'$, and hence we have $\langle x, y' \rangle = \langle x, z' + y' \rangle \neq 0$, which shows that the dual system $(N, N')$ is separated. Therefore we have

$$\dim(N) \leqq 2^{\dim(N')}. \tag{3}$$

Thus, combining (1), (2), and (3), we have the desired conclusion.

Now suppose that the dimension of $N$ is finite. Then in the proof of the case where $\dim(N)$ is infinite, we have

$$\dim(N') = \dim(N) \tag{4}$$

instead of the inequality (3). Thus we have $\dim(N) \leqq \operatorname{codim}(M')$ from (1), (2), and (4).

REMARK. More generally, theorem 6 is valid for linear mappings $u$ of $E$ onto $M$ satisfying the following condition:

(*) $\quad u(M) = M$ and $u^{-1}(0) \frown M = \{0\}$.

In fact, for every linear mapping $u$ of $E$ into $M$ satisfying the condition (*), we have $u^{-1}(0) + M = E$; let $p$ be the projection of $E$ onto $M$ such that $p^{-1}(0) = u^{-1}(0)$; then we have $(u|_M) \circ p = u$, where $u|_M$ denotes the restriction of $u$ to $M$. Let $\tau_1$ be the weakest topology on $M$ which makes $u|_M$ continuous as a mapping onto $M$ with the topology $\tau_0$, and let $\tau$ be a locally convex topology on $E$. Then since $u = (u|_M) \circ p$, the mapping $u$ is continuous for the topologies $\tau$ and $\tau_0$ if and only if $p$ is continuous for the topologies $\tau$ and $\tau_1$. Furthermore, the dual of $M$ for the topology $\tau_1$ is isomorphic to the dual of $M$ for the topology $\tau_0$. Therefore, the above mentioned statement follows from Theorem 6.

As a corollary of Theorem 6, we have the following

THEOREM 5′. *Let $M$ be a vector subspace of an infinite dimensional vector space $E$. If $\dim(E) \leqq 2^{\alpha}$, where $\alpha = 2^{\dim(M)}$, then for every algebraic supplement $N$ of $M$ in $E$, there exists a locally convex Hausdorff topology on $E$ which makes $M$ dense in $E$ and $N$ closed.*

Proof. Let $B$ be a base of $M$. For each $x \in B$, we define a linear functional $x'$ on $M$ by setting, for every $y \in B$,

$$\langle y, x' \rangle = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Denote by $M'$ the vector subspace of $M^*$ spanned by the set $\{x'; x\in B\}$. We have then $\dim(M')=\dim(M)<2^{\dim(M)}=\dim(M^*)$. Therefore $\operatorname{codim}(M')=\dim(M^*)$, and hence we have by the assumption $\operatorname{codim}(M)\leqq\dim(E)\leqq 2^{\mathfrak{a}}=2^{\dim(M^*)}=2^{\operatorname{codim}(M')}$. Since $\operatorname{codim}(M')$ is infinite, applying Theorem 6 for the weak topology $\sigma(M, M')$, we have the conclusion.

The following corollary is a consequence of Theorem 3.

COROLLARY. *Let $M$ be an infinite dimensional vector subspace of a vector space $E$. Then for every vector subspace $F\supseteq M$ of dimension $\leqq 2^{\mathfrak{a}}$, where $\mathfrak{a}=2^{\dim(M)}$, and for every algebraic supplement $N$ of $M$ in $F$, there exists a locally convex Hausdorff topology on $E$ for which we have $\overline{M}=F$ and $\overline{N}=N$.*