34. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XV

By Sakuji INOUE
Faculty of Education, Kumamoto University
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Let $N_j, D_j (j=1,2,3,\cdots,n)$, $\{\lambda_\nu\}_{\nu=1,2,3,\ldots}$, $f_{1\alpha}$, $f_{2\alpha}$, $f'_{1\alpha}$, $f'_{2\alpha}$, $g_{j\beta}$, $g'_{j\beta}$, and $T(\lambda)$ be the same notations as those defined in Part XIII (cf. Proc. Japan Acad., Vol. 40, No. 7, 492–493 (1964)), and let $R(\lambda)$ be the ordinary part of $T(\lambda)$. Then

$$T(\lambda)=R(\lambda)+\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha}(f_{1\alpha}+f_{2\alpha}),$$
$$(f'_{1\alpha}+f'_{2\alpha}))+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta}g_{j\beta},\ g'_{j\beta}),$$

and $T(\lambda)$ possesses the properties (i), (ii), and (iii) described in Part XIII. Analytically speaking, the first principal part of $T(\lambda)$ is given by

$$\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha}f_{1\alpha},\ f'_{1\alpha})=\sum_{\alpha=1}^{m}\sum_{\nu=1}^{\infty}\frac{c_\alpha^{(\nu)}}{(\lambda-\lambda_\nu)^\alpha},$$

where if $\lambda_1=\lambda_2=\cdots=\lambda_{m_1}$, $\lambda_{m_1+1}=\lambda_{m_1+2}=\cdots=\lambda_{m_2}$, and so on, then $\sum_{\nu=1}^{\infty}\frac{c_\alpha^{(\nu)}}{(\lambda-\lambda_\nu)^\alpha}$ means the sum

$$\frac{c_\alpha^{(1)}}{(\lambda-\lambda_1)^\alpha}+\frac{c_\alpha^{(m_1+1)}}{(\lambda-\lambda_{m_1+1})^\alpha}+\cdots=\frac{c_\alpha^{(m_1)}}{(\lambda-\lambda_{m_1})^\alpha}+\frac{c_\alpha^{(m_2)}}{(\lambda-\lambda_{m_2})^\alpha}+\cdots,$$

as will be seen by the definition of $c_\alpha^{(\nu)}$ in the above-mentioned paper; and in addition, the second principal part of $T(\lambda)$ is given by

$$\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha}f_{2\alpha},\ f'_{2\alpha})+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta}g_{j\beta},\ g'_{j\beta})$$
$$=\sum_{\alpha=1}^{m}\int_{\Omega\cup D_1}\frac{1}{(\lambda-z)^\alpha}d(K^{(1)}(z)f_{2\alpha},\ f'_{2\alpha})+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}\int_{D_j}\frac{1}{(\lambda-z)^\beta}d(K^{(j)}(z)g_{j\beta},\ g'_{j\beta}),$$

where Ω denotes the set of all those accumulation points of $\{\lambda_\nu\}_{\nu=1,2,3,\ldots}$ which do not belong to $\{\lambda_\nu\}$ itself and $\{K^{(j)}(z)\}$ is the complex spectral family associated with the bounded normal operator N_j $(j=1,2,3,\cdots,n)$. These facts are clear from the respective definitions of the notations $f_{1\alpha}$, $f_{2\alpha}$, $f'_{1\alpha}$, $f'_{2\alpha}$, $g_{j\beta}$, $g'_{j\beta}$, $c_\alpha^{(\nu)}$, N_j, and D_j.

Since, by definition, $\{\lambda_\nu\}$ is an arbitrarily prescribed bounded set of denumerably infinite complex numbers, we may and do suppose here that it is everywhere dense on an open rectifiable Jordan curve Γ; and as a special case, we consider the function $\hat{T}(\lambda)$ defined by

$$\text{(A)}\quad \hat{T}(\lambda)=R(\lambda)+\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha}f_{1\alpha},\ f'_{1\alpha})+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta}g_{j\beta},\ g'_{j\beta}).$$

Then it is obvious that every λ_ν is a pole of $\hat{T}(\lambda)$ in the sense of the functional analysis (but not in the sense of the classical theory of functions) and that every point on Γ is a singularity of $\hat{T}(\lambda)$. In this paper we shall investigate the question as to whether Picard's theorem for essential singularities in the classical theory of functions can be valid for $\hat{T}(\lambda)$ in a suitably small neighbourhood of any point on Γ and shall discuss the number of Picard's exceptional values for a suitably small domain containing Γ in the interior of itself.

Theorem 41. Let $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$ be a bounded set of denumerably infinite points which are everywhere dense on an open rectifiable Jordan curve Γ in the complex λ-plane; let $\hat{T}(\lambda)$ be the function defined by (A); let ξ be an arbitrarily given point on Γ; let δ_ξ be a positive number less than the distance from ξ to the set $\bigcup_{j=1}^{n} D_j$; let $\mathfrak{D}_\xi$ be the domain $\{\lambda: |\lambda-\xi|<\delta_\xi\}$; and let Δ_{δ_ξ} be the domain $\mathfrak{D}_\xi-(\mathfrak{D}_\xi\cap\Gamma)$. Then in Δ_{δ_ξ} $\hat{T}(\lambda)$ assumes every finite value, with the possible exception of at most two finite values, an infinite number of times.

Proof. Let A and B denote the extremities of Γ; let M_1 be the middle point of the segment $\widehat{A\xi}$ of Γ; and let M_2 be the middle point of the segment $\widehat{M_1\xi}$ of Γ. By repeating this procedure we obtain the infinite sequence of points M_n, $(n=1, 2, 3, \cdots)$, tending to ξ on the segment $\widehat{A\xi}$ of Γ; and similarly we can assign another infinite sequence of points M'_n, $(n=1, 2, 3, \cdots)$, tending to ξ on the segment $\widehat{B\xi}$ of Γ. On the assumption that p is any poitive integer greater than a suitably large positive integer G, we now denote by λ_{p_1} the point with the least value of indices ν in the set $\{\lambda_\nu\}_{\nu=p,p+1,p+2,\dots}\cap\widehat{AM_1}$ and similarly by λ_{p_2} the point with the least value of indices ν in the set $\{\lambda_\nu\}_{\nu=p,p+1,p+2,\dots}\cap\widehat{M_1M_2}$. By repeating this procedure we have an infinite sequence of points $\{\lambda_{p_\kappa}\}_{\kappa=1,2,3,\dots}$ tending to ξ on the segment $\widehat{A\xi}$ of Γ. In the same manner we can assign another infinite sequence of points $\{\lambda_{p'_\kappa}\}_{\kappa=1,2,3,\dots}$ tending to ξ on the segment $\widehat{B\xi}$ of Γ. We next consider in connection with $f_{1\alpha}=\sum_{\nu=1}^{\infty} a_{1\alpha}^{(\nu)}\varphi_\nu^{(1)}\in\mathfrak{M}_1$ the element

$$f_{1\alpha}^{(p)}=\sum_{\nu=1}^{p-1} a_{1\alpha}^{(\nu)}\varphi_\nu^{(1)}+\sum_{\kappa=1}^{\infty} a_{1\alpha}^{(p_\kappa)}\varphi_{p_\kappa}^{(1)}+\sum_{\kappa=1}^{\infty} a_{1\alpha}^{(p'_\kappa)}\varphi_{p'_\kappa}^{(1)}\in\mathfrak{M}_1,$$

where $\{\varphi_\nu^{(1)}\}_{\nu=1,2,3,\dots}$ is such an incomplete orthonormal system in the complete and separable Hilbert space $\mathfrak{H}$ of infinite dimension as was defined at the beginning of Part XIII and $\mathfrak{M}_1$ denotes the subspace determined by $\{\varphi_\nu^{(1)}\}$, and then construct the infinite sequence of auxiliary functions

$$\text{(B)}\quad \hat{T}_p(\lambda)=R(\lambda)+\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha}f_{1\alpha}^{(p)}, f'_{1\alpha})$$
$$+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta}g_{j\beta}, g'_{j\beta}),\ (p=G+1, G+2, G+3, \cdots),$$

for the function $\hat{T}(\lambda)$ given by (A).

Suppose now that E_{δ_ξ} is an arbitrary closed domain with simple closed boundary in Δ_{δ_ξ} and that λ is a point belonging to E_{δ_ξ}. Then

$$|\hat{T}(\lambda)-\hat{T}_p(\lambda)|=\left|\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha}(f_{1\alpha}-f_{1\alpha}^{(p)}),\, f'_{1\alpha})\right|$$
$$\leqq \sum_{\alpha=1}^{m}\|(\lambda I-N_1)^{-1}\|^{\alpha}\,\|f_{1\alpha}-f_{1\alpha}^{(p)}\|\,\|f'_{1\alpha}\|$$
$$(p=G+1,\ G+2,\ G+3,\ \cdots),$$

where

$$\|f_{1\alpha}-f_{1\alpha}^{(p)}\|=\{\textstyle\sum_{p\leq\nu\neq p_\kappa, p'_\kappa(\kappa=1,2,3,\cdots)}|a_{1\alpha}^{(\nu)}|^2\}^{\frac{1}{2}}.$$

Since $\sup_{\lambda\in E_{\delta_\xi}}\|(\lambda I-N_1)^{-1}\|^{\alpha}$ is finite because of the fact that any point λ in E_{δ_ξ} is a point of the resolvent set of every N_j by virtue of the definition of E_{δ_ξ}, and since $\sum_{\nu=1}^{\infty}|a_{1\alpha}^{(\nu)}|^2<\infty$, by choosing in advance G sufficiently large for an arbitrarily given small positive ε we can find from the just established inequality that $|\hat{T}(\lambda)-\hat{T}_p(\lambda)|<\varepsilon$ holds for all $\lambda\in E_{\delta_\xi}$ and every positive integer p greater than G and hence that the infinite sequence of functions $\{\hat{T}_p(\lambda)\}_{p\geqq G+1}$ regular in E_{δ_ξ} converges uniformly to $\hat{T}(\lambda)$ in E_{δ_ξ} itself. Consequently, according to a well-known theorem based on the Rouché theorem, in the interior of E_{δ_ξ} the number of zero-points of every $\hat{T}_p(\lambda)$ with $G'\leqq p<\infty$ for an appropriately chosen positive integer G' greater than G equals that of zero-points of $\hat{T}(\lambda)$. As a result, it is easily found that for any finite complex number ω and a suitably large positive integer P, every $\hat{T}_p(\lambda)$ with $P\leqq p<\infty$ and $\hat{T}(\lambda)$ have in the interior of E_{δ_ξ} the same number (inclusive of 0) of ω-points. On the other hand, since every $\hat{T}_p(\lambda)$ with $P\leqq p<\infty$ is meromorphic in the domain $\mathfrak{D}'\{\lambda\colon 0<|\lambda-\xi|<\delta_\xi\}$ and has denumerably infinite points $\{\lambda_\nu\}_{\nu=1,2,3,\cdots,p-1}$ and $\{\lambda_{p_\kappa},\lambda_{p'_\kappa}\}_{\kappa=1,2,3,\cdots}$ tending to ξ as its poles in the sense of the classical theory of functions, it assumes in $\mathfrak{D}'$ every value, with the possible exception of at most two values, an infinite number of times by virtue of Picard's theorem in the wider sense; and here the exceptional values in the sense of Picard are finite, and ξ is the unique accumulation point of ω-points of $\hat{T}_p(\lambda)$ on the assumption that ω is not its exceptional value. In addition, if we denote by Δ an arbitrary bounded domain which contains Γ in the interior of itself but not any point of $\bigcup_{j=1}^{n}D_j$, then $\hat{T}(\lambda)$ has every point on Γ as its singularity and is regular in $\Delta-(\Delta\cap\Gamma)$, as can be seen from its expression (A) and the hypothesis that both $f_{1\alpha}$ and $f'_{1\alpha}$ are elements consisting of all $\varphi_\nu^{(1)}$. Hence any ω-point of $\hat{T}(\lambda)$ in $\mathfrak{D}'$ lies on Δ_{δ_ξ} but not on Γ. Choosing now an arbitrary not exceptional value ω for one of the functions $\hat{T}_p(\lambda)$ with $P\leqq p<\infty$ and supposing, contrary to what we wish to prove, that the number of ω-points of $\hat{T}(\lambda)$ is finite in Δ_{δ_ξ}, it follows that in Δ_{δ_ξ}

the number of ω-points of any function belonging to the family $\{\hat{T}_p(\lambda)\}_{p\geqq P}$ would be finite. In fact, the number of ω-points of any $\hat{T}_p(\lambda)\,(p\geqq P)$ in $\varDelta_{\delta_\xi}$ coincides with that of ω-points of $\hat{T}(\lambda)$ in $\varDelta_{\delta_\xi}$, as will be found from the fact that the respective numbers of ω-points of these two functions are identical in the interior of an arbitrary closed domain E_{δ_ξ} contained in the open domain $\varDelta_{\delta_\xi}$. Accordingly we attain to the result that denumerably infinite ω-points of any $\hat{T}_p(\lambda)$ $(p\geqq P)$ would lie on Γ. On the other hand, since

$$\hat{T}(\lambda)-\hat{T}_p(\lambda)=\sum_{\alpha=1}^{m}\int_{\{\lambda_\nu\}\cup\Omega}\frac{1}{(\lambda-z)^\alpha}d(K^{(1)}(z)(f_{1\alpha}-f_{1\alpha}^{(p)}),\,f'_{1\alpha})$$
$$=\sum_{\alpha=1}^{m}\int_{\{\lambda_\nu\}}\frac{1}{(\lambda-z)^\alpha}d(K^{(1)}(z)(f_{1\alpha}-f_{1\alpha}^{(p)}),\,f'_{1\alpha}),$$

and since $|(K^{(1)}(z)(f_{1\alpha}-f_{1\alpha}^{(p)}),\,f'_{1\alpha})|\leqq\{\sum_{p\leqq\nu\neq p_\kappa,p'_\kappa(\kappa=1,2,3,\cdots)}|a_{1\alpha}^{(\nu)}|^2\}^{\frac{1}{2}}\,||f'_{1\alpha}||\to 0$, irrespective of both of λ and z, as $p\to\infty$, in the entire complex λ-plane (inclusive of the point at infinity) $\hat{T}(\lambda)$ is the limit function of the family $\{\hat{T}_p(\lambda)\}_{p\geqq P}$. Since, moreover, every point on Γ is a singularity of $\hat{T}(\lambda)$, the above result to which we attained is absurd. Consequently the supposition that the number of ω-points of $\hat{T}(\lambda)$ is finite in $\varDelta_{\delta_\xi}$ must be rejected.

The theorem has thus been proved.

Remark. More generally, the result of Theorem 41 holds for the case where the set $\{\lambda_\nu\}$ is everywhere dense on a finite number of open or closed rectifiable Jordan curves, as will be seen by minor modifications of the method used above.

Theorem 42. Let $\hat{T}(\lambda)$, Γ, ξ, δ_ξ, and $\varDelta_{\delta_\xi}$ be the same notations as those defined in the statement of Theorem 41, and $\varDelta$ an open domain covered by $\varDelta_{\delta_\xi}$ as ξ ranges over Γ. Then the number of finite exceptional values of $\hat{T}(\lambda)$ for $\varDelta$ is at most two.

Proof. Suppose that with respect to $\varDelta_{\delta_\xi}$, $\hat{T}(\lambda)$ has a finite exceptional value ω_ξ associated with ξ and that ζ is an arbitrary point, not ξ, on $\Gamma\cap\bar{\varDelta}_{\delta_\xi}$. Then, with respect to an open domain $\varDelta_{\delta_\zeta}$ defined for ζ in the same way as $\varDelta_{\delta_\xi}$ was defined for ξ, ω_ξ is also an exceptional value associated with ζ of $\hat{T}(\lambda)$, so that ω_ξ is an exceptional value of $\hat{T}(\lambda)$ for the extended domain $\varDelta_{\delta_\xi}\cup\varDelta_{\delta_\zeta}$. By iterating this procedure finite times we can attain to the result that ω_ξ is an exceptional value of $\hat{T}(\lambda)$ for such extended open domain $\varDelta$ as was defined in the statement of the present theorem. In addition, since, by Theorem 41, $\hat{T}(\lambda)$ has at most two finite exceptional values for $\varDelta_{\delta_\xi}$ (even if they exist), it is at once obvious that in respect of $\varDelta$, the number of finite exceptional values of $\hat{T}(\lambda)$ in the sense of Picard cannot exceed two.

Thus the proof of theorem is complete.

Remark. With small modifications the same technique may be used to show that Theorem 42 is also extended to the more general case stated in the preceding remark.

Correction to Sakuji Inoue: "Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XII" (Proc. Japan Acad., 40, 487–491 (1964)).

Page 489, line 1: For "$\cdots + \sum_{\mu=1}^{\infty} \bar{\Psi}_{\mu}(\lambda) \otimes L_{\Phi_{\mu}(\lambda)}$", read "$\cdots + \sum_{\mu=1}^{\infty} \bar{\Psi}_{\mu}(\lambda) \otimes L_{\hat{\varphi}_{\mu}(\lambda)}$".