28. *Subsemigroups of Completely 0-Simple Semigroups. I*[*)]

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1. Introduction. A completely 0-simple semigroup S is isomorphic to a regular Rees matrix semigroup over a group with zero $G^0=G\cup\{0\}$ with sandwich matrix $P=(p_{ji})$, $p_{ji}\in G^0$, $i\in L_0$, $j\in M_0$ where each row and each column of P contains at least one non-zero element [1, 2, 3]. That is to say,

$$S=\{0\}\cup\{(x;\, i, j)\mid x\in G,\, i\in L_0,\, j\in M_0\}$$

where the multiplication is defined as follows:

$$0\cdot(x;\, i, j)=(x;\, i, j)\cdot 0=0\cdot 0=0 \quad \text{for all } (x;\, i, j)$$

$$(x;\, i, j)\cdot(y;\, k, l)=\begin{cases}0 & \text{if } p_{jk}=0,\\ (xp_{jk}y;\, i, l) & \text{if } p_{jk}\neq 0.\end{cases}$$

G is called the structure group of S.

It is known that any subsemigroup of a finite complete 0-simple semigroup S is completely 0-simple if P contains no zero [1, Ex. 19, p. 85]. This is not true for the general case without assumption of finiteness. Actually the type of subsemigroups of completely 0-simple semigroups is the generalization of completely 0-simple semigroups. The purpose of this series of the papers is to determine all subsemigroups of completely 0-simple semigroups. However, as the first step towards this study, the present paper treats 0-simple subsemigroups of completely 0-simple semigroups in the special case where G^0 is finite. In such a case, any subsemigroup of S is completely 0-simple, or simple, if P contains no zero; any 0-simple subsemigroup of S is completely 0-simple if P contains zero. Also we discuss how to construct such subsemigroups in a given S. We remark that the discussions in the case where P contains no zero includes those in the case where S is completely simple [1, 2] since, if S is a completely simple semigroup and if S^0 denotes a completely 0-simple semigroup such that $S^0=S\cup\{0\}$, then any subsemigroup of S^0 containing 0 is a subsemigroup of S with 0 adjoined.

The detailed proof will be published elsewhere.

2. Support. We start with subsemigroups of a completely 0-simple semigroup R in which the structure group G of R is the

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identity 1 alone, i.e., the case where the elements of the sandwich matrix P are either 1 or 0. R is characterized as follows:
Let $R=\{0\}\cup R'$ where $R'=\{(i,j)\mid i\in L_0, j\in M_0\}=L_0\times M_0$.
Let Δ denote a subset of the set $M_0\times L_0$ satisfying the following condition: for each $i\in L_0$ there is at least one $j\in M_0$ such that $(j,i)\in\Delta$; for each $j\in M_0$ there is at least one $i\in L_0$ such that $(j,i)\in\Delta$. Define a binary operation on R as follows:

$$0\cdot(i,j)=(i,j)\cdot 0=0\cdot 0=0 \qquad \text{for all } (i,j)$$

$$(i,j)\cdot(k,l)=\begin{cases}0 & \text{if } (j,k)\notin\Delta\\ (i,l) & \text{if } (j,k)\in\Delta\end{cases}$$

Let V be a subsemigroup of R, and let $\Gamma=V\cap\Delta^{-1}$ where $\Delta^{-1}=\{(i,j)\mid(j,i)\in\Delta\}$. If Γ is empty, then V is a nullsemigroup [1]. We may assume $\Gamma\neq\phi$. Now we define an equivalence relation $\bar{\rho}$ on Γ by the transitive closure of a relation ρ on Γ:

$$(i,j)\rho(k,l) \text{ if and only if } i=k \text{ or } j=l.$$

Thus Γ is divided into equivalence classes modulo $\bar{\rho}$:

$$\Gamma=\bigcup_{\alpha\in J}\Gamma_\alpha.$$

Further define

$$L_\alpha=\{i\mid(i,j)\in\Gamma_\alpha \text{ for some } j\in M_0\},\qquad L=\bigcup_{\alpha\in J}L_\alpha,$$

$$M_\alpha=\{j\mid(i,j)\in\Gamma_\alpha \text{ for some } i\in L_0\},\qquad M=\bigcup_{\alpha\in J}M_\alpha.$$

Lemma 1. *Let V be a subsemigroup of R. Then we have*

(1) $L_\alpha\times M_\alpha\subseteq V$ *for all* $\alpha\in J$.

(2) $(L_\alpha\times M_\beta)\cap V\neq\phi$, $\alpha,\beta\in J$, *implies* $L_\alpha\times M_\beta\subseteq V$.

Lemma 2. *If V is a 0-simple subsemigroup of R, then*

$$V\subseteq(L\times M)\cup\{0\}.$$

With using these lemmas, we have

Theorem 1. *If $\Delta^{-1}=R'$, then V is a subsemigroup of R which contains 0 if and only if $V=\{0\}\cup(L\times M)$. If $\Delta^{-1}\neq R'$, then a subsemigroup V of R is 0-simple if and only if $V=\{0\}\cup(L\times M)$.*

The semigroup R is closely related to a general case as follows: Let S be a completely 0-simple semigroup over finite G^0 with sandwich matrix $P=(p_{ji})$, $j\in M_0$, $i\in L_0$; let $R=\{0\}\cup(L_0\times M_0)$.

Define a binary operation on R by

$$0\cdot x=x\cdot 0=0$$

$$(i,j)\cdot(k,l)=\begin{cases}0 & \text{if } p_{jk}=0\\ (i,l) & \text{if } p_{jk}\neq 0.\end{cases}$$

Then R is a homomorphic image of S under the mapping φ:

$$0\varphi=0,\qquad (z;i,j)\varphi=(i,j).$$

Definition. Let T be a subsemigroup of S containing 0. The restriction $T\varphi$ of φ to T is called the support of T. If there are two subsets L and M of L_0 and M_0 respectively such that $T\varphi=$

$\{0\}\cup(L\times M)$, then T is said to have a rectangular support.

3. Subsemigroups of S. In the consequence of Theorem 1, we can say that if T is a 0-simple subsemigroup of S then T has a rectangular support. We shall derive the converse of this statement. For this purpose, some preparation is needed.

Let S be a completely 0-simple semigroup over finite G^0, and T a subsemigroup of S. Define several notations:

$S_{ij}=\{(x;i,j)\mid x\in G\}$, and hence $S=\{0\}\cup(\bigcup_{i\in L_0, j\in M_0} S_{ij})$

$T_{ij}=T\cap S_{ij}$

$\Gamma=\{(i,j)\mid p_{ji}\neq 0, \text{ and } T_{ij}\neq\phi\}$

$L=\{i\mid (i,j)\in\Gamma,\ T_{ij}\neq\phi \text{ for some } j\in M_0\}$

$M=\{j\mid (i,j)\in\Gamma,\ T_{ij}\neq\phi \text{ for some } i\in L_0\}$

$G_{ij}=\{x\mid (x;i,j)\in T_{ij}\}$

Lemma 2. *Let $(i,j)\in\Gamma$. If T is a subsemigroup of S then $G_{ij}p_{ji}$ and $p_{ji}G_{ij}$ are isomorphic onto the subsemigroup T_{ij}. If G is finite, then they are subgroups of G, and $G_{ij}p_{ji}=G_{ik}p_{ki}$ and $p_{ji}G_{ij}=p_{jk}G_{kj}$ for all $(i,j),(i,k),(k,j)\in\Gamma$.*

Because of Lemma 2, we define

$$G_{i\cdot}=G_{ij}p_{ji},\quad G_{\cdot j}=p_{ji}G_{ij} \quad \text{whenever } (i,j)\in\Gamma.$$

By Theorem 1 we have the following theorem, generalization of Theorem 1.

Theorem 2. *If $\Delta^{-1}=R'$, any subsemigroup with zero T of S over finite G^0 has a rectangular support. If $\Delta^{-1}\neq R'$, then T is 0-simple if and only if T has a rectangular support.*

The following lemma is useful for the arguments later.

Lemma 3. *If G is finite and if T is a subsemigroup with 0 of S, then, for each $(i,j)\in L\times M$ with $G_{ij}\neq\phi$, there is an element $a\in G$ such that*

$$G_{ij}=aG_{\cdot j}=G_{i\cdot}a.$$

Let T be a 0-simple subsemigroup with support $\{0\}\cup(L\times M)$. For each $(i,j)\in L\times M$, a matrix $\Pi=(\pi_{ji})$, $j\in M$, $i\in L$, is defined as follows:

$$\pi_{ji}=\begin{cases} p_{ji} & \text{if } p_{ji}\neq 0 \\ a_{ji}^{-1} & \text{if } p_{ji}=0\end{cases}$$

where a_{ji} is one of the elements satisfying $G_{ij}=a_{ji}G_{\cdot j}=G_{i\cdot}a_{ji}$ in Lemma 3. *Thus*

$$G_{i\cdot}=G_{ij}\pi_{ji},\quad G_{\cdot j}=\pi_{ji}G_{ij},\quad (i,j)\in L\times M.$$

We have the construction theorem.

Theorem 3. *Let S be a completely 0-simple semigroup over finite G^0, that is, a regular Rees matrix semigroup over finite G^0 with sandwich $(M_0\times L_0)$-matrix (p_{ji}), $j\in M_0$, $i\in L_0$. To construct a subsemigroup with zero T of S:*

(1) *Choose subsets* L *and* M *of* L_0 *and* M_0 *respectively such that for any* $i \in L$ *there is a* $p_{ji} \neq 0$, *and for any* $l \in M$ *there is a* $p_{lk} \neq 0$.

(2) *For each* $(i, j) \in L \times M$, π_{ji} *is defined to be an element of* G *such that*

$$\pi_{ji} = \begin{cases} p_{ji} & \text{if } p_{ji} \neq 0. \\ \text{arbitrary} & \text{if } p_{ji} = 0. \end{cases}$$

(3) *For a fixed* $j = 1 \in M$, *there is at least a subgroup* $G_{\cdot 1}$ *of* G *satisfying*

$$\pi_{1i}\pi_{ji}^{-1}\pi_{j1}\pi_{11}^{-1} \in G_{\cdot 1} \quad \text{for all} \quad i \in L,\ j \in M.$$

(4) *Let* $G_{i\cdot} = \pi_{1i}^{-1} G_{\cdot 1} \pi_{1i}$, $G_{\cdot j} = \pi_{ji} G_{i\cdot} \pi_{ji}^{-1}$,

$$G_{ij} = \pi_{ji}^{-1} G_{\cdot j}.$$

(5) $T = \left(\bigcup_{i \in L, j \in M} T_{ij} \right) \cup \{0\}$ *where* $T_{ij} = \{(x; i, j) \mid x \in G_{ij}\}$.

Then T *is a* 0-*simple subsemigroup of* S.

Conversely any 0-*simple subsemigroup of* S *is obtained in this way.*

Theorem 4. T *is isomorphic with the regular Rees matrix semigroup over* $G_{\cdot 1} \cup \{0\}$ *with sandwich matrix* (s_{ji}), $i \in L$, $j \in M$, *where* s_{ji} *is defined as follows*:

$$s_{ji} = \begin{cases} \pi_{11}\pi_{j1}^{-1}\pi_{ji}\pi_{1i}^{-1} & \text{if } p_{ji} \neq 0 \\ 0 & \text{if } p_{ji} = 0 \end{cases}$$

References

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