26. The Relation between (N, p_n) and $(\bar{N}, p_n)$ Summability

By Kazuo ISHIGURO
Department of Mathematics, Hokkaido University, Sapporo
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We suppose, throughout this note, that

$$p_n>0, \quad \sum_{n=0}^{\infty} p_n=\infty,$$
$$P_n=p_0+p_1+\cdots+p_n, \quad n=0, 1, \cdots.$$

The Nörlund transformation (N, p_n) is defined as transforming the sequence $\{s_n\}$ into the sequence $\{t_n\}$ by means of the equation

$$t_n=\frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu}s_\nu . \tag{1}$$

As is well known, this transformation is regular if

$$\lim_{n\to\infty}\frac{p_n}{P_n}=0. \tag{2}$$

See Hardy [1], p. 64.

The discontinuous Riesz transformation $(\bar{N}, p_n)$ is defined as transforming the sequence $\{s_n\}$ into the sequence $\{u_n\}$ by means of the equation

$$u_n=\frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu . \tag{3}$$

This transformation is regular (see Hardy [1], p. 57).

As is easily seen, the transformations (N, p_n) and $(\bar{N}, p_n)$ take symmetric forms, hence we can expect the close relation between them. We shall prove here the following

Theorem 1. *Suppose that*

(4) $\{p_n\}$ *is non-increasing,*

and that

(5) $p_n\geq\sigma>0, \quad n=0, 1, \cdots.$

Then $(\bar{N}, p_n)$ *implies**) (N, p_n).

Proof. From (3) we have

$$s_n=\frac{P_nu_n-P_{n-1}u_{n-1}}{p_n}, \quad n=0, 1, \cdots,$$

with $P_{-1}=u_{-1}=0$. Hence, from (1),

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*) Given two summability methods A, B, we say that A *implies* B if any sequence summable A is summable B to the same sum.

$$(6)\qquad t_n=\frac{1}{P_n}\sum_{\nu=0}^{n}\left\{\frac{p_{n-\nu}}{p_\nu}P_\nu u_\nu-\frac{p_{n-\nu}}{p_\nu}P_{\nu-1}u_{\nu-1}\right\}$$
$$=\frac{1}{P_n}\sum_{\nu=0}^{n-1}\left\{\frac{p_{n-\nu}}{p_\nu}-\frac{p_{n-\nu-1}}{p_{\nu+1}}\right\}P_\nu u_\nu+\frac{p_0}{p_n}u_n$$
$$=\sum_{\nu=0}^{n}a_{n\nu}u_\nu,$$

where

$$(7)\qquad a_{n\nu}=\frac{P_\nu}{P_n}\left\{\frac{p_{n-\nu}}{p_\nu}-\frac{p_{n-\nu-1}}{p_{\nu+1}}\right\}\quad \text{for}\quad \nu=0,1,\cdots n,$$

with $p_{-1}=0$.

Now if $s_\nu=1$ for all ν, then $t_n=1$, $u_n=1$ for all n. Hence $\sum_{\nu=0}^{n}a_{n\nu}=1$ for all n. Also, since $P_n\to\infty$ and $\{p_n\}$ is bounded, it is clear that $a_{n\nu}\to 0$ as $n\to\infty$ for any fixed ν. Hence a necessary and sufficient condition for the transformation (6) to be regular is that

$$\sum_{\nu=0}^{n}|a_{n\nu}|=O(1).$$

Since, from (4),

$$\frac{p_n}{p_0}\leq\frac{p_{n-1}}{p_1}\leq\cdots\leq\frac{p_1}{p_{n-1}}\leq\frac{p_0}{p_n},$$

we have

$$\sum_{\nu=0}^{n}|a_{n\nu}|=-\sum_{\nu=0}^{n-1}a_{n\nu}+\frac{p_0}{p_n}$$
$$=-\frac{1}{P_n}\sum_{\nu=0}^{n-1}P_\nu\left\{\frac{p_{n-\nu}}{p_\nu}-\frac{p_{n-\nu-1}}{p_{\nu+1}}\right\}+\frac{p_0}{p_n}$$
$$=-\frac{1}{P_n}\left\{P_0\frac{p_n}{p_0}-P_{n-1}\frac{p_0}{p_n}+\right.$$
$$\left.+\sum_{\nu=1}^{n-1}\frac{p_{n-\nu}}{p_\nu}(P_\nu-P_{\nu-1})\right\}+\frac{p_0}{p_n}$$
$$=-\frac{1}{P_n}(p_n+p_{n-1}+\cdots+p_1)+\frac{P_{n-1}}{P_n}\frac{p_0}{p_n}+\frac{p_0}{p_n}$$
$$\leq\frac{2p_0}{\sigma}$$

from (7) and (5). This proves our assertion.

From the proof of our theorem, we obtain the following

Corollary. *If*

$$\inf_n p_n=0,$$

then $(\bar{N}, p_n)$ does not imply (N, p_n).

Next we shall prove the following

Theorem 2. *Suppose that*

(8) $\{p_n\}$ *is non-decreasing,*

and that

$$(2) \qquad \lim_{n\to\infty} \frac{p_n}{P_n} = 0.$$

Then $(\bar{N}, p_n)$ *implies* (N, p_n).

Proof. As in the proof of Theorem 1, we get $\sum_{\nu=0}^{n} a_{n\nu} = 1$ for all n. Next we see easily, from (2), that $a_{n\nu} \to 0$ as $n \to \infty$ for any fixed ν. Finally we have, from (8),

$$\frac{p_n}{p_0} \geq \frac{p_{n-1}}{p_1} \geq \cdots \geq \frac{p_1}{p_{n-1}} \geq \frac{p_0}{p_n},$$

hence

$$\begin{aligned}\sum_{\nu=0}^{n} |a_{n\nu}| &= \sum_{\nu=0}^{n} a_{n\nu} \\ &= \frac{1}{P_n}(p_n + p_{n-1} + \cdots + p_1) - \frac{P_{n-1}}{P_n}\frac{p_0}{p_n} + \frac{p_0}{p_n} \\ &\leq 2.\end{aligned}$$

Collecting the above estimations we obtain the desired conclusion.

Reference

[1] G. H. Hardy: Divergent Series. Oxford (1949).