47. On the Convergence Theorem for Star-shaped Sets in E^n

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Introduction. It is well known, as Blaschke convergence theorem, that a uniformly bounded infinite collection of closed convex sets in a finite dimensional Minkowski space contains a sequence which converges to a non-empty compact convex set. The convergence problem for star-shaped sets seems open up to-day (cf. [1]).

In this paper, modifying F. A. Valentine's proof of the Blaschke convergence theorem in [1], we prove a convergence theorem for star-shaped sets in the n-dimensional euclidean space E^n. In the case of E^3, Z. A. Melzak's result [2] is known.

1. Notations and lemmas. In the following, we consider sets in the n-dimensional euclidean space E^n only.

Let S be a star-shaped set relative to a point p. Then the closure of S, denoted by clS, is a star-shaped set relative to the point p. If $\{S^\alpha;\ \alpha\in \text{index set}\}$ is a finite or an infinite collection of star-shaped sets relative to a point p, then $\bigcup_\alpha S^\alpha$ and $\bigcap_\alpha S^\alpha$ are star-shaped relative to the point p.

An *ε-parallel set* A_ε of a set A is defined by

$$A_\varepsilon\equiv \bigcup_{a\in A} K(a,\ \varepsilon),\ (0\leqq\varepsilon,\ \varepsilon\in \text{reals}),$$

where $K(a,\ \varepsilon)$ denotes the solid sphere with center a and radius ε. The distance between the two points x and y is denoted by $d(x,\ y)$.

Lemma 1. $(A_\rho)_\sigma\subset A_{\rho+\sigma}$.

Proof. Let x be a point in $(A_\rho)_\sigma$. Then there is a point $y\in A_\rho$ such that $d(x,\ y)\leqq\sigma$. Similarly there is a point $z\in A$ such that $d(y,\ z)\leqq\rho$. Hence we have

$$d(x,\ z)\leqq d(x,\ y)+d(y,\ z)=\sigma+\rho.$$

Therefore x is a point of $A_{\rho+\sigma}$.

The distance $d(A,\ B)$ between the two sets A and B is defined by

$$d(A,\ B)=\inf_{\substack{A\subset B_\rho\\ B\subset A_\rho}} \rho.$$

If A and B degenerate to two points x and y, the distance function coincides with the ordinary distance of E^n.

Lemma 2. *A collection of compact sub-sets becomes a metric space with the metric defined above.*

Proof. i) $d(A, A)=0$,

ii) $d(A, B)=d(B, A)$,

and iii) $d(A, B)>0$, if $A\neq B$

are trivial consequences of the definition and the compactness of the sets A and B. To prove

iv) $d(A, C)\leqq d(A, B)+d(B, C)$,

let $d(A, B)=\rho$, $d(B, C)=\sigma$ and $\rho+\sigma=\tau$. Then $B\subset A_\rho$, and by lemma 1, $B_\sigma\subset(A_\rho)_\sigma\subset A_\tau$. Since $C\subset B_\sigma$, we have $C\subset A_\tau$. Similarly we have $A\subset C_\tau$. Hence $d(A, C)\leqq\tau=d(A, B)+d(B, C)$.

A family of sets $\mathfrak{M}=\{A^\alpha;\ \alpha\in$ index set$\}$ is uniformly bounded if there exists a solid sphere $K(O, R)$ with center at the origin and with radius $R(0\leqq R<\infty)$ which contains the entire sets of $\mathfrak{M}$.

Given a set A and a point p, the set

$$_pA\equiv\{x|^\exists y\in A,\ x=\beta p+\gamma y \text{ for } \beta\geqq 0,\ \gamma\geqq 0 \text{ and } \beta+\gamma=1\}$$

is called the *star extension* of a set A relative to a point p. It is easily seen that, if a set A is compact then $_pA$ is also compact.

Lemma 3. *If a set A is star-shaped relative to a point p, and q be a point such that $d(p, q)<\varepsilon$, then $d(A, {}_qA)<\varepsilon$.*

Proof. By the definition of star extension, $A\subset{}_qA\subset{}_qA_\varepsilon$. If $x\in{}_qA$, then there is a point $a\in A$ such that $x=\beta q+\gamma a$, $\beta\geqq 0$, $\gamma\geqq 0$ and $\beta+\gamma=1$. Let $y=\beta p+\gamma a$. Then $y\in A$, for A is star-shaped relative to p; and

$$d(x, y)=\|x-y\|=\|\beta q+\gamma a-\|\beta p-\gamma a\|$$
$$=\beta\|q-p\|<\beta\cdot\varepsilon\leqq\varepsilon.$$

Hence $d(A, {}_qA)<\varepsilon$.

A sequence of sets $\{A^i;\ i=1, 2, \cdots\}$ is said to converge to a set A if $\lim_{i\to\infty} d(A^i, A)=0$.

2. Convergence Theorem.

Theorem. *Let $\mathfrak{M}=\{S^\alpha;\ \alpha\in$ index set$\}$ be a uniformly bounded infinite collection of compact star-shaped sets in E^n. Then $\mathfrak{M}$ contains a sequence which converges to a non-empty compact star-shaped set.*

Proof. By the same reasoning of [1] (Th. 3.8), we can prove that there exists a sequence $\{S^n;\ n=1, 2, \cdots\}$ such that for any $\varepsilon>0$ there is a number N and for any $m>N$ and $n>N$, we have $d(S^m, S^n)<\varepsilon$.

Now each S^n is a star-shaped set, so let p^n be a point relative to which S^n is star-shaped. Since $\{p^n;\ n=1, 2, \cdots\}$ is contained in the solid sphere $K(O, R)$, which also contains the entire sets of $\mathfrak{M}$, the infinite sequence $\{p^n\}$ has a convergent sub-sequence $\{p^{n_i}\}$. Let $\lim_{i\to\infty} p^{n_i}=p$.

Let us now denote n_i as n. Then we can say by lemma 3 that there exists a sequence $\{(S^n, p^n); n=1, 2, \cdots\}$, such that for any $\varepsilon>0$ we have

$$d(p^n, p)<\varepsilon \text{ and } d(S^m, S^n)<\varepsilon \text{ for any } m>N,\ n>N. \qquad (1)$$

Let C^n be the star extension of S^n relative to the point p, then by Lemma 2, Lemma 3 and (1) we have,

$$d(C^n, S^n)<\varepsilon, \text{ for } n>N, \qquad (2)$$

and

$$d(C^m, C^n)\leqq d(C^m, S^m)+d(S^m, S^n)+d(S^n, C^n)<3\varepsilon, \text{ for } m>N,\ n>N. \qquad (3)$$

Let $B^n\equiv cl(C^n\cup C^{n+1}\cup\cdots)\subset K(O, R)$

and

$$S\equiv\bigcap_{n=1}^{\infty} B^n. \qquad (4)$$

Since $(C^n\cup C^{n+1}\cup\cdots)$ is star-shaped relative to the point p, B^n is compact and star-shaped relative to the point p. Moreover $B^{n+1}\subset B^n$. Therefore S is a non-empty compact and star-shaped set relative to the point p.

The convergence of $\{S^n\}$ to the limit S is proved similar to [1], on account of (1), (3), and (4). Let S_ε and C^n_ε be the ε-parallel sets of S and of C^n respectively (where $d(S_\varepsilon, S)=\varepsilon$ and $d(C^n_\varepsilon, C^n)=\varepsilon$). Given any $\varepsilon>0$, there is a number N' such that for any $n>N'$ we have $B^n\subset S_\varepsilon$. For if not so, $B^n\cap\partial(S_\varepsilon)\neq\phi$*) for infinitely many $n>N'$, and B^n's are compact and $B^{n+1}\subset B^n$. Therefore we have $S\cap\partial(S_\varepsilon)\neq\phi$, which is a contradiction. Hence

$$C^n\subset B^n\subset S_\varepsilon\subset S_{3\varepsilon} \text{ for } n>N'. \qquad (5)$$

The condition (3) implies $C^m\subset C^n_{3\varepsilon}$ for $m>N$, $n>N$, and by (3) and definition of B^n we have $B^n\subset C^n_{3\varepsilon}$ for $n>N$. Hence

$$S\subset B^n\subset C^n_3 \text{ for } n>N. \qquad (6)$$

Therefore by (5) and (6)

$$d(S, C^n)<3\varepsilon \text{ for } n>\max(N, N'). \qquad (7)$$

By Lemma 2, (2) and (7), we have

$$d(S^n, S)\leqq d(S^n, C^n)+d(S, C^n)<4\varepsilon.$$

Hence we have proved that $\lim_{n\to\infty} S^n=S$. This completes the proof.

References

[1] F. A. Valentine: Convex sets. Mcgraw-Hill (1964).
[2] Z. A. Melzak: A class of star-shaped bodies. Can. Math. Bull., **2**, 175-180 (1959).

*) ∂ denotes "the boundary of". ϕ denotes the empty set.