59. *Resolvent Kernels on a Martin Space*

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Let R be a Green space, M be its Martin boundary and μ be the harmonic measure on M relative to a fixed point of R. As a result of the author's previous paper [3], we can see that, if every point of M is an exit boundary point of R,

$$D(u, u)=\int_M\int_M (u(\xi)-u(\eta))^2 U(\xi, \eta)\mu(d\xi)\mu(d\eta) \tag{1}$$

represents the Dirichlet integral on R, up to a constant, of the harmonic function with boundary value $u(\xi)$, $\xi\in$ M, where $U(\xi, \eta)$ is Feller's kernel (cf. Doob [2]).

We shall apply this fact to form a system of resolvent kernels on $(R\cup M)\times R$ which dominate on $R\times R$ the resolvent kernels of a Brownian motion on R. As the generalized normal derivatives of the potentials defined by these kernels, we may have zero function on M. The construction of these kernels is our main purpose.

To this aim, we shall first define a system of operators R^α, $\alpha>0$ on $L^2(\mu)$ such that, for every $\varphi\in L^2(\mu)$, $R^\alpha\varphi$ satisfies

$$D(R^\alpha\varphi, v)+2\int_M\int_M R^\alpha\varphi(\xi)U_\alpha(\xi, \eta)v(\eta)\mu(d\xi)\mu(d\eta)=2\int_M \varphi(\xi)v(\xi)\mu(d\xi) \tag{2}$$

for any v in a certain function class, where U_α is α-order Feller's kernel. Next, we shall prove the positivity of $R^\alpha(\alpha>0)$ and the continuity of $R^\alpha\varphi$ in a certain sense. Finally, using $\{R^\alpha, \alpha>0\}$, we shall form resolvent kernels satisfying the properties cited above.

§1. Positive operators $R^\alpha(\alpha>0)$ on $L^2(\mu)$.

Let $p(t, x, y)$ $t>0$, $x, y\in$ R be the transition function of a Brownian motion on R. Its resolvent kernel is defined by

$$G_\alpha(x, y)=\int_0^{+\infty} e^{-\alpha t}p(t, x, y)dt,\ \alpha>0,\ x, y\in \mathrm{R}.$$

For the Martin K-function $K(x, \xi)$, put

$$K_\alpha(x, \xi)=K(x, \xi)-\alpha\int_R G_\alpha(x, y)K(y, \xi)dy, \qquad x\in \mathrm{R},\ \xi\in \mathrm{M},\ \alpha>0.$$

We call $\xi\in$ M an exit boundary point if and only if $K_\alpha(x, \xi)\neq 0$ for some $x\in$ R and $\alpha>0$. For $\xi, \eta\in$ M, $\alpha>0$, $U_\alpha(\xi, \eta)=\alpha\int_R K(x, \xi)K_\alpha(x, \eta)dx$ is monotone increasing in α and we call $U(\xi, \eta)=\lim\limits_{\alpha\to+\infty} U_\alpha(\xi, \eta)$ Feller's kernel (cf. [3]).

From now on, we assume that

(A.1) almost every (μ) point of M is exit,

$$\text{(A.2)}\qquad \int_{\mathrm{M}}\int_{\mathrm{M}} U_\alpha(\xi, \eta)\mu(d\xi)\mu(d\eta) < +\infty \quad \text{for some } \alpha>0.$$

We note that (A.1) and (A.2) hold when R is a bounded domain of the N-dimensional Euclidean space.

Now put $U_\alpha(u, u) = \int_{\mathrm{M}}\int_{\mathrm{M}} u(\xi) U_\alpha(\xi, \eta) u(\eta)\mu(d\xi)\mu(d\eta)$,

$D_\alpha(u, u) = D(u, u) + 2U_\alpha(u, u)$ and $|||u|||_\alpha = \sqrt{D_\alpha(u, u)}$, $\alpha>0$.

We consider a function space

$$\tilde{\mathcal{D}} = \{u : D(u, u) < +\infty,\ U_\alpha(|u|, |u|) < +\infty\}.$$

$\tilde{\mathcal{D}}$ is independent of $\alpha(>0)$ and forms a Hilbert space with the inner product

$$D_\alpha(u, v) = \frac{1}{4}\{|||u+v|||_\alpha^2 - |||u-v|||_\alpha^2\},\ \alpha>0.$$

$\int_{\mathrm{M}} U_\alpha(\xi, \eta)u(\eta)\mu(d\eta)$ will be denoted by $U_\alpha u(\xi)$, $\xi\in \mathrm{M}$, $\alpha>0$.

Easily we see that

$$(3)\qquad |||u|||_\alpha^2 = \int_{\mathrm{M}}\int_{\mathrm{M}} (u(\xi)-u(\eta))^2[U(\xi, \eta) - U_\alpha(\xi, \eta)]\mu(d\xi)\mu(d\eta) + 2\int_{\mathrm{M}} u(\xi)^2 U_\alpha 1(\xi)\mu(d\xi) \qquad u\in\tilde{\mathcal{D}}.$$

Since $U(\xi, \eta) \geqq U_\alpha(\xi, \eta)$, $\xi, \eta\in \mathrm{M}$, $\alpha>0$ and $\inf_{\xi\in M} U_\alpha 1(\xi)$ is strictly positive, the formula (3) leads us to the following

Lemma 1. (i) *$\tilde{\mathcal{D}}\subset L^2(\mu)$, and for any $u\in\tilde{\mathcal{D}}$, $\|u\|_{L^2(\mu)} \leqq \delta(\alpha)|||u|||_\alpha$, where $\delta(\alpha)^2 = (2\inf_{\xi\in M} U_\alpha 1(\xi))^{-1}$.*

(ii) *If $u\in\tilde{\mathcal{D}}$ and v is a contraction of u* (cf. [1]), *that is, $|v(\xi)|\leqq|u(\xi)|$ and $|v(\xi)-v(\eta)|\leqq|u(\xi)-u(\eta)|$ for any $\xi, \eta\in \mathrm{M}$, then $v\in\tilde{\mathcal{D}}$ and $|||v|||_\alpha \leqq |||u|||_\alpha$.*

It follows from Lemma 1 (i) that, for $\varphi\in L^2(\mu)$, there exists a function of $\tilde{\mathcal{D}}$ uniquely (denoted by $R^\alpha\varphi$) such that the equation (2) holds for any $v\in\tilde{\mathcal{D}}$. We can also associate $R^\alpha\varphi\in\tilde{\mathcal{D}}$ with the function $\varphi = U_\beta w$, $w\in\tilde{\mathcal{D}}$, $\beta>0$. Lemma 1 (ii) assures the positivity of R^α (cf. [1]).

Theorem 1. (i) $R^\alpha\varphi\geqq 0$ a.e(μ) *if* $\varphi\geqq 0$ a.e(μ) ($\alpha>0$).

(ii) $R^\alpha U_\alpha 1 = 1$ a.e(μ) ($\alpha>0$).

(iii) *For* $\varphi\in L^2(\mu)$, $\alpha>0$, $\beta>0$,

$$R^\alpha\varphi - R^\beta\varphi + R^\alpha(U_\alpha - U_\beta)R^\beta\varphi = 0 \qquad \text{a.e}(\mu).$$

§ 2. The quasi continuity of $R^\alpha\varphi$.

Let $\alpha>0$ be fixed throughout this section. For every positive integer m, we define $D^m(u, u)$ by

$$D^m(u, u) = \int_{\mathrm{M}}\int_{\mathrm{M}} (u(\xi)-u(\eta))^2 U_m(\xi, \eta)\mu(d\xi)\mu(d\eta).$$

For $m\geqq\alpha$ put $D_\alpha^m(u, u) = D^m(u, u) + 2U_\alpha(u, u)$, $|||u|||_\alpha^m = \sqrt{D_\alpha^m(u, u)}$ and

$$\tilde{\mathscr{D}}'=\{u : D^m(u, u)<+\infty,\ U_\alpha(|u|, |u|)<+\infty\}.$$

$\tilde{\mathscr{D}}'$ is independent of m and forms a Hilbert space with the inner product $D^m_\alpha(u, v)=\frac{1}{4}((|||u+v|||^m_\alpha)^2-(|||u-v|||^m_\alpha)^2)$. For $\varphi\in L^2(\mu)$, there exists an unique element $R^\alpha_m\varphi$ of $\tilde{\mathscr{D}}'$ such that

$$D^m_\alpha(R^\alpha_m\varphi, v)=2\int_M\varphi(\xi)v(\xi)\mu(d\xi)$$

holds for any $v\in\tilde{\mathscr{D}}'$. We denote by C the totality of continuous functions on M.

Lemma 2. (i) *If* $U_\beta(C)\subset C$ *for any* $\beta>0$, *then* $R^\alpha_m(C)\subset C$.

(ii) $R^\alpha_m\varphi$ *converges to* $R^\alpha\varphi$ *in the following sense.*

$$|||R^\alpha_m\varphi-R^\alpha\varphi|||^m_\alpha\to 0 \quad as\ m\to+\infty,\ \varphi\in L^2(\mu).$$

Now for the open set $\mathrm{E}\subset\mathrm{M}$, we define $C^m_\alpha(\mathrm{E})$ by $C^m_\alpha(\mathrm{E})=\inf_{\substack{u\in\tilde{\mathscr{D}}'\\ u>1 \text{ on } \mathrm{E}}} D^m_\alpha(u, u)$. It follows from the analogous formula to (3) for D^m_α, that for the continuous function u on M and $\varepsilon>0$, the inequality

(4) $$C^m_\alpha\{x : |u|>\varepsilon\}\leqq\frac{D^m_\alpha(u, u)}{\varepsilon^2}$$ holds. (cf. [1]).

(4) and Lemma 2 imply the next

Theorem 2. (*the quasi continuity of* $R^\alpha\varphi, \varphi\in C$). *Let* φ *be continuous. If* $U_\beta(C)\subset C$ *for every* $\beta>0$, *then, for any integer* m ($\geqq\alpha$), *there exists an increasing sequence of closed subsets* E_k ($k=1, 2, \cdots$) *of* M *such that* $\lim_{k\to+\infty}C^m_\alpha(\mathrm{M}-\mathrm{E}_k)=0$ *and* $R^\alpha\varphi$ *is continuous on* E_k ($k=1, 2, \cdots$).

§ 3. Resolvent kernels on $(\mathrm{R}\cup\mathrm{M})\times\mathrm{R}$.

For every $y\in\mathrm{R}$, $\varphi(\cdot)=K_\alpha(y, \cdot)$, $\alpha>0$, is a boundary function of $L^2(\mu)$. We shall define $R_\alpha(x, y)$, $\alpha>0$, on $(\mathrm{R}\cup\mathrm{M})\times\mathrm{R}$ as follows: For $\xi\in\mathrm{M}$, $y\in\mathrm{R}$, $R_\alpha(\xi, y)=R^\alpha\varphi(\xi)$, where $\varphi(\cdot)=K_\alpha(y, \cdot)$. For $x\in\mathrm{R}$, $y\in\mathrm{R}$, $R_\alpha(x, y)=G_\alpha(x, y)+\int_{\mathrm{M}}K_\alpha(x, \xi)R_\alpha(\xi, y)\mu(d\xi)$.

Consider the function $u_1(x)=Hu(x)+u_p(x)$, where Hu is a BLD harmonic function with a fine boundary function u and u_p is a potential of a measure ν on R. We assume that every BLD harmonic function Hv with a fine boundary function $v\in\tilde{\mathscr{D}}$ is integrable on R with respect to the absolute variation of ν.

We call φ is a generalized normal derivative of u_1 on M, if $\int_{\mathrm{M}}\varphi(\xi)v(\xi)\mu(d\xi)<+\infty$ and $D(u, v)=-2\int_{\mathrm{M}}\varphi(\xi)v(\xi)\mu(d\xi)+2\int_{\mathrm{R}}Hv(x)\nu(dx)$ for any $v\in\tilde{\mathscr{D}}$ (cf. [2]).

Theorem 3. *Let* y *be an arbitrarily fixed point of* R. *Then, for every* $x\in\mathrm{R}$ *and* a.e(μ) $x\in\mathrm{M}$,

(i) $R_\alpha(x, y)\geqq 0$, $\alpha>0$,

(ii) $R_\alpha(x, y) - R_\beta(x, y) + (\alpha - \beta)\int_{\mathrm{R}} R_\alpha(x, z) R_\beta(z, y) dz = 0,\ \alpha, \beta > 0.$

Further, (iii) $\alpha \int_{\mathrm{R}} R_\alpha(x, y) dy = 1$, *for every* $x \in \mathrm{R}$ *and* a.e(μ) $x \in \mathrm{M}$.

(iv) *If f is a bounded function with a compact support on R, then* $R_\alpha f(x) = \int_{\mathrm{R}} R_\alpha(x, y) f(y) dy$ *has the zero function as its generalized normal derivative on M.*

(v) *If f is a continuous function with a compact support on* R, *then* $\lim_{\alpha \to +\infty} \alpha R_\alpha f(x) = f(x) \qquad x \in \mathrm{R}$.

Remark. To $R_\alpha(x, y)$, $\alpha > 0$, $x, y \in \mathrm{R}$, there corresponds a conservative Markov process on R. Such a method of the construction of resolvent kernels is applicable to the case in which R, dx, and $G_\alpha(x, y)$ are, respectively, the state space, the excessive measure and the symmetric Green function of the more general Markov process than an absorbing barrier Brownian motion.

References

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