57. On Tannaka's Conjecture on the Cohomologically Trivial Modules

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(Comm. by Kenjiro SHODA, M.J.A., April 12, 1965)

Let G be a finite group. A G-module A is called cohomologically trivial when we have $H^q(H, A)=0$ for all integers q and all subgroups H of G. It is known that A is cohomologically trivial for the following cases:

(I) (Nakayama [1]) $H^r(H, A)=H^{r+1}(H, A)=0$ for some integer r and for all subgroups H of G.

(II) (Nakayama [1], [2]) G is a p-group, and $H^r(G, A)=H^{r+1}(G, A)=0$ for some integer r.

(III) (Nakayama [2]) G is a p-group, A is a G-module such that $pA=0$, and $H^r(G, A)=0$ for some integer r.

T. Tannaka proved that we could replace $r, r+1$ by $r, r+s$ in the first case, where s is a fixed odd integer (Th. 1). He also conjectured that the same would be true in the second case. Here we consider this problem in some special cases, while the general case is still open.

I must express my hearty thanks to Professor Tannaka, who suggested me this problem and has encouraged me during the preparation of this paper.

1. We first recall the fundamental exact sequences. Let G be a finite group and H a normal subgroup of G. If A is a G-module, we denote by A^H the set of H-invariant elements, and by A_H the factor module of A by I_HA, where I_HA is a submodule of A generated by the elements of the form $(\sigma-1)a$ for σ in H and a in A. N_H denotes a norm map with respect to H, and ${}_{N_H}A$ denotes the kernel of N_H in A. We have then the following exact sequences

(1) $0\to H^1(G/H, A^H)\to H^1(G, A)\to H^1(H, A)^G\to H^2(G/H, A^H)\to H^2(G, A)$

(2) $0\to H^0(G/H, N_HA)\to H^0(G, A)\to H^0(H, A)^G\to H^1(G/H, N_HA)\to H^1(G, A)$

(3) $0\to H^{-1}(G/H, A_H)\to H^{-1}(G, A)\to H^{-1}(H, A)^G\to H^0(G/H, A_H)\to H^0(G, A)$

(4) $0\leftarrow H^0(G/H, A^H)\leftarrow H^0(G, A)\leftarrow H^0(H, A)_G\leftarrow H^{-1}(G/H, A^H)\leftarrow H^{-1}(G, A)$

(5) $0\leftarrow H^{-1}(G/H, N_HA)\leftarrow H^{-1}(G, A)\leftarrow H^{-1}(H, A)_G\leftarrow H^{-2}(G/H, N_HA)\leftarrow H^{-2}(G, A)$

(6) $0 \leftarrow H^{-2}(G/H, A_H) \leftarrow H^{-2}(G, A) \leftarrow H^{-2}(H, A)_G \leftarrow H^{-3}(G/H, A_H)$
$\leftarrow H^{-3}(G, A).$

In (1), if $H^1(H, A) = \cdots = H^{q-1}(H, A) = 0$, we can replace suffices 1 and 2 by q and $q+1$ respectively, and similarly for others. Relations between (1), (2), and (3) (or (4), (5), and (6)) are as follows: Let I be the augmentation ideal of the group ring $Z[G]$ of G over the ring of integers. Then the exact sequence

(7) $0 \to A \otimes I \to A \otimes Z[G] \to A \to 0$

induces the exact sequences

(8) $0 \to (A \otimes I)^H \to (A \otimes Z[G])^H \to N_H A \to 0$

and

(9) $0 \to A \otimes I /_{N_H}(A \otimes I) \to A \otimes Z[G] /_{N_H}(A \otimes Z[G]) \to A/I_H A \to 0.$

Now in (1), replacing A by $A \otimes I$, we have for instance,

$$\begin{array}{ccc} 0 \longrightarrow H^1(G/H, (A\otimes I)^H) & \longrightarrow & H^1(G, A\otimes I) \\ \uparrow \delta & & \uparrow \delta \\ H^0(G/H, N_H A) & \dashrightarrow & H^0(G, A) \end{array}.$$

As $(A \otimes Z[G])^H$ is G/H-regular, two δ's are isomorphisms. So we can transfer (1) to (2). As $N_H A \cong A/_{N_H}A$ and $N_H(A \otimes Z[G]) = (A \otimes Z[G])^H$, (9) is also written in the form

(9′) $0 \to N_H(A \otimes I) \to (A \otimes Z[G])^H \to A_H \to 0.$

Now replacing A by $A \otimes I$ in (2), we transfer exact sequence (2) to (3) as before. By this transference, (4) in [3] corresponds to (6) obtained from the homological fundamental exact sequence dual to (1).

2. In this section we prove Tannaka's theorem and his conjecture in two special cases.

Theorem 1. (Tannaka) Let G be a finite group, and A a G-module. Let s be an odd integer. Then if $H^r(H, A) = H^{r+s}(H, A) = 0$ for some integer r and for all subgroups H of G, A is cohomologically trivial.

Proof. The theorem is trivial if G is a cyclic group. We use induction on the order of G. By induction assumption, H-module A is cohomologically trivial if H is a proper subgroup of G. In particular, if G is not a p-group, G_p-module A is cohomologically trivial for all p-Sylow subgroups G_p of G, where p runs over all prime numbers dividing the order of G. Then G-module A is cohomologically trivial. So we may assume that G is a p-group. Let H be a maximal subgroup of G. As $H^q(H, A) = 0$ for all integers q, we have exact sequences

$$0 \to H^{r+s}(G/H, A^H) \to H^{r+s}(G, A) = 0$$
$$0 \to H^{r+1}(G/H, A^H) \to H^{r+1}(G, A) \to H^{r+1}(H, A) = 0.^{1)}$$

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1) We reverse the direction of the arrows, if necessary.

As G/H is cyclic, $H^{r+s}(G/H, A^H)=0$ means $H^{r+1}(G/H, A^H)=0$. Therefore we have $H^{r+1}(G, A)=0$. Together with $H^r(G, A)=0$ we have the G-module A is cohomologically trivial by (II).[2)]

Tannaka's conjecture: Let G be a p-group, A a G-module. If $H^r(G, A)=H^{r+2s+1}(G, A)=0$ for some integers r and s, A is cohomologically trivial.

From now on, G always denotes a p-group, and A a G-module. We frequently use the dimension shifting by Artin-Tate dimension shifters I and J, and note that if A is finitely generated (resp. finite), so also are $A\otimes I$ and $A\otimes J$. Therefore we can shift dimensions arbitrarily without changing these conditions.

Theorem 2. Let A be finitely generated, and $H^r(G, A)=H^{r+3}(G, A)=0$. Then A is cohomologically trivial.

Proof. By dimension shifting we may assume $H^1(G, A)=H^{-2}(G, A)=0$. Let H be a normal subgroup of G such that G/H is cyclic. By exact sequences (1) and (6), we have $H^1(G/H, A^H)=H^{-2}(G/H, A_H)=0$. The map $N_H: A_H\to A^H$ induces exact sequences of G/H-modules

$$0\to H^{-1}(H, A)\to A_H\to A^H\to H^0(H, A)\to 0.$$

As $H^{-1}(H, A)$ and $H^0(H, A)$ are finite and G/H is cyclic, we have by the well known relation between Herbrand's quotients

$$h_{2/1}(G/H, A_H)=h_{2/1}(G/H, A^H).$$

Then from $H^1(G/H, A^H)=H^{-2}(G/H, A_H)=0$, it follows that $H^2(G/H, A^H)=H^{-3}(G/H, A_H)=0$. Coming back to (1) and (6) again, we have $H^1(H, A)^G=H^{-2}(H, A)_G=0$. As the augmentation ideal $\bar{I}=I\otimes(Z/p^nZ)$ in $(Z/p^nZ)[G]$ is nilpotent, it follows that $H^1(H, A)=H^{-2}(H, A)=0$. By induction on the order of G, H-module A may be assumed to be cohomologically trivial, $H^0(H, A)=0$ in particular. As $H^0(G/H, A^H)=0$, $H^0(G, A)=0$ follows from (4). Being $H^1(G, A)=H^0(G, A)=0$, A is cohomologically trivial by (II).

Theorem 3. Let G be a direct product of two cyclic subgroups. Then Tannaka's conjecture holds.

Proof. This follows from the following well known lemma.

Lemma. Let G be a finite abelian group, H its subgroup. Then the restriction map Res: $H^2(G, Z)\to H^2(H, Z)$ is surjective.

By dimension shifting, we have G-modules B and C such that $H^q(g, B)\cong H^{q+r+2s}(g, A)$, $H^q(g, C)\cong H^{q+r}(g, A)$ for all subgroups g of G, and for all integers q. Then by assumption

$$H^{-2s}(G, B)=H^1(G, B)=0$$
$$H^0(G, C)=H^{2s+1}(G, C)=0$$

hold. Let H be one of the direct factors of G. Then as H is cyclic,

2) Theorem 1 can also be proved without employing the theorem (II).

$H^2(H, Z) \cong H^0(H, Z)$ is also cyclic. Let $\alpha \in H^2(H, Z)$ be one of its generators, and $\beta \in H^2(G, Z)$ be such that $\operatorname{Res}\beta = \alpha$, which exists by the lemma. By the relations between the cup product and the maps Res and Inj (injection or transfer map), the diagrams

$$\begin{array}{ccc} H^{q+2}(G, B) & \xleftarrow{\text{Inj}} & H^{q+2}(H, B) \\ \uparrow{\scriptstyle \cup\beta} & & \uparrow{\scriptstyle \cup\alpha} \\ H^q(G, B) & \xleftarrow{\text{Inj}} & H^q(H, B) \end{array}$$

$$\begin{array}{ccc} H^{q+2}(G, C) & \xrightarrow{\text{Res}} & H^{q+2}(H, C) \\ \uparrow{\scriptstyle \cup\beta} & & \uparrow{\scriptstyle \cup\alpha} \\ H^q(G, C) & \xrightarrow{\text{Res}} & H^q(H, C) \end{array}$$

are commutative for all integers q, and the cup products by α give isomorphisms. Proceeding in the same way, we have commutative diagrams

$$\begin{array}{ccc} H^0(G, B) & \xleftarrow{\text{Inj}} & H^0(H, B) \\ \uparrow & & \uparrow \\ 0 = H^{-2s}(G, B) & \xleftarrow{\text{Inj}} & H^{-2s}(H, B) \end{array}$$

$$\begin{array}{ccc} 0 = H^{2s+1}(G, C) & \xrightarrow{\text{Res}} & H^{2s+1}(H, C) \\ \uparrow & & \uparrow \\ H^1(G, C) & \xrightarrow{\text{Res}} & H^1(H, C). \end{array}$$ [3]

As the vertical arrows on the right hand side are isomorphisms, we conclude that the maps $\text{Inj} : H^0(H, B) \to H^0(G, B)$ and $\text{Res} : H^1(G, C) \to H^1(H, C)$ are zero maps. By the exact sequences

$$0 \to H^1(G/H, B^H) \to H^1(G, B) = 0$$
$$0 \leftarrow H^0(G/H, C^H) \leftarrow H^0(G, C) = 0,$$

we have $H^1(G/H, B^H) = H^0(G/H, C^H) = 0$. As G/H is also cyclic, $H^{-1}(G/H, B^H) = H^2(G/H, C^H) = 0$. By the exact sequences (1) and (4) used for C and B in place of A, we have $H^1(H, C)^G = H^0(H, B)_G = 0$. As in the proof of the theorem 2, we have $H^1(H, C) = H^0(H, B) = 0$. Coming back to the relations between the cohomology groups of A and that of B and C, we have

$$H^{r+2s}(H, A) = H^{r+1}(H, A) = 0.$$

As H is cyclic, we have $H^q(H, A) = 0$ for all q. So, the sequences

$$0 \to H^{r+2s+1}(G/H, A^H) \to H^{r+2s+1}(G, A) = 0$$
$$0 \to H^{r+1}(G/H, A^H) \to H^{r+1}(G, A) \to H^{r+1}(H, A) = 0$$

are exact.[4] We have $H^{r+2s+1}(G/H, A^H) = H^{r+1}(G/H, A^H) = 0$ from the first sequence, and the second shows $H^{r+1}(G, A) = 0$. By the assumption

3) We may assume $s > 0$.

4) Cf. footnote 1)

of the theorem, we have $H^r(G, A)=0$. Therefore the theorem follows from (II).

3. Now we consider an analogue to (III), that is:

Theorem 4. Let G be a finite p-group, and A a finite G-module such that $H^r(G, A)=0$ holds for some integer r. Then A is cohomologically trivial.

Proof. By the remark preceding the theorem 2, we can use dimension shifting and assume that $H^1(G, A)=0$. If G is cyclic, the theorem is trivial by Herbrand's lemma. So, by induction on the order of G, we may assume that the finite g-module B is cohomologically trivial, if g is a p-group of the order smaller than that of G, and $H^1(g, B)=0$. Let H be a proper normal subgroup of G. Then by the sequence (1), we have $H^1(G/H, A^H)=0$. By induction assumption, $H^q(G/H, A^H)=0$ holds for any integer q. Again by (1), we have $H^1(H, A)^G=0$ and then $H^1(H, A)=0$. Also by induction assumption $H^0(H, A)=0$, and the sequence (4) shows that $H^0(G, A)=0$ holds. By (II), we have then the result.

Remark. In general, (III) is not true even for the bounded modules A, i.e. such modules that $nA=0$ for some integer n. For example, let p be an odd prime, and A be an additive group having $\{a_1, a_2, \cdots\}$ as a basis over Z/p^2Z. Let G be a cyclic group of order p, having σ as a generator. A can be made a G-module by defining as $\sigma a_i = a_i + p a_{i+1}$. Then we can easily show that $A^G = NA = {}_N A = pA$ and $IA = pA'$, taking A' as a submodule of A generated by $\{a_2, a_3, \cdots\}$. This shows that $H^0(G, A)=0$ but $H^{-1}(G, A)\neq 0$.

References

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