86. Notes on (m, n)-Ideals. III

By Sándor LAJOS

K. Marx University, Budapest, Hungary

(Comm. by Kinjirô KUNUGI, M.J.A., May 19, 1965)

The first two papers of this series are [2] and [3].

Let S be a semigroup. An (m, n)-ideal of S is called *locally minimal* if it contains no proper (m, n)-ideal. If a semigroup S contains no proper (m, n)-ideal, where m, n are arbitrary fixed positive integers, then by Theorem 4, S is a group. Thus we have the following result.

Theorem 10. *The locally minimal (m, n)-ideals of a semigroup S are groups. (m, n are arbitrary positive integers.)*

In case of $m=n=1$, Theorem 10 gives the

Corollary. *The locally minimal bi-ideals in a semigroup S are groups.*

An (m, n)-ideal A of a semigroup S is called *minimal*, if it does not properly contain any (m, n)-ideal of S. We prove the

Theorem 11. *Any locally minimal (m, n)-ideal of a semigroup S is also a minimal (m, n)-ideal of S.*

Proof. Let S be a semigroup, A a locally minimal (m, n)-ideal of S. If B would be an (m, n)-ideal of S, which is properly contained in A, then by Theorem 1, B would be an (m, n)-ideal of the semigroup A, because of $B=A\cap B$. But A has no proper (m, n)-ideal, thus A is indeed minimal (m, n)-ideal of S.

We shall call an (m, n)-ideal of a semigroup S *universally minimal* in S, if it is contained in every (m, n)-ideal of S. Obviously, the universally minimal (m, n)-ideal of S is also minimal. Such an universally minimal (m, n)-ideal of S is uniquely determined, as easy to see. Concerning universally minimal (m, n)-ideal of a semigroup S we prove the

Theorem 12. *Let S be a semigroup having a two-sided ideal G, which is at the same time a subgroup of S. Then G is the universally minimal (m, n)-ideal of S. (m, n are arbitrary non-negative integers.)*

Proof. Suppose that S is a semigroup having a two-sided ideal G, which is a subgroup of S. Then G is an (m, n)-ideal of S, for any non-negative integers m, n. Let A be an arbitrary (m, n)-ideal of S. Then

$$A^m G A^n \subseteq A^m S A^n \subseteq A.$$

On the other hand, the set A^mGA^n is an (m, n)-ideal of G. Hence by Theorem 4, it follows that

$$A^mGA^n=G.$$

Therefore G is contained in any (m, n)-ideal of S, that is, G is the universally minimal (m, n)-ideal of S.

Theorem 12 contains the following statement (see [4]).

Corollary. *Let S be a semigroup containing a subgroup G, which is a two-sided ideal of S. Then G is the universally minimal left (right, two-sided) ideal of S.*

A semigroup S is called a *homogroup* (or a semigroup having zeroid elements, see [1]), if

(i) S contains an idempotent e;

(ii) For each $a \in S$ there exists an element a' in S so that $aa'=e$;

(iii) $ea=ae$, for every $a \in S$.

Such an idempotent e satisfying the conditions (i), (ii), and (iii) is uniquely determined. It is easy to see that a semigroup having zero is a homogroup. Thierrin [5] has proved that every finite commutative semigroup is also a homogroup. It is also known that the set $eS=Se$ in a homogroup S is a group, which is a two-sided ideal of S. Conversely, if a semigroups has a subgroup G, which is a two-sided ideal of S, then S is a homogroup. Thus a semigroup S is a homogroup if, and only if, S contains a subgroup, which is two-sided ideal of S.

By the Theorem 12, *the group-ideal of a homogroup H is the universally minimal (m, n)-ideal of H.* Now we prove the

Theorem 13. *Any (m, n)-ideal of a homogroup H is also a homogroup.*

Proof. Let H be a homogroup containing G as group-ideal, and let A be an arbitrary (m, n)-ideal of H. Then

$$A^mGA^n \subseteq A \cap G,$$

and thus the intersection $A \cap G$ is not vacuous. On the other hand, $A \cap G$ is an (m, n)-ideal of G, by Theorem 1, and hence

$$A \cap G=G,$$

because of a group has no proper (m, n)-ideal. Therefore the semigroup A contains G as group-ideal, and thus A is a homogroup.

An easy consequence of the Theorem 13 is the following

Corollary. *Any left (right, two-sided) ideal of a homogroup H is also a homogroup.*

References

[1] A.H. Clifford and D.D. Miller: Semigroups having zeroid elements. Amer. J. Math., **70**, 117–125 (1948).
[2] S. Lajos: Notes on (m, n)-ideals. I. Proc. Japan Acad., **39**, 419–421 (1963).
[3] ——: Notes on (m, n)-ideals, II. Proc. Japan Acad., **40**, 631–632 (1964).
[4] E. S. Ljapin: Semigroups (in Russian). Moscow (1960).
[5] G. Thierrin: Sur les homogroupes. C. R. Acad. Sci. Paris, **234**, 1519–1521 (1952).