82. *Remarks on a Continuous Kernel*

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1. Introduction. Choquet and Deny [1, 2] proved the following theorem: A strictly positive continuous kernel $V: C_K \to C$ satisfies the balayage principle if and only if it satisfies the domination principle. (For the notations and the definitions see Section 2.) In the present note we show that a continuous kernel $V: C \to C$ satisfies the balayage principle on any open set if and only if it satisfies the domination principle under the assumption that $V(C)$ is dense in C. In Section 4 we show that if a continuous kernel $V: C \to C$ satisfies the two conditions, the denseness of $V(C)$ in C and the complete maximum principle, then it is a continuous kernel of Hunt.

2. Notations and definitions. Let X be a locally compact Hausdorff space, and B denote the Borel field on X. Let $C=C(X)$ be the totality of bounded continuous real valued functions on X. C is a real Banach space with the norm $\|f\|=\sup_{x\in X}|f(x)|$. Let $C_K=C_K(X)$ be the totality of continuous real valued functions on X with compact support. Let $M=M(X)$ and $M_K=M_K(X)$ be the totalities of real Radon measures on X and of those with compact support, respectively. We denote by $C^+, \cdots$ the subsets of the above sets consisting of positive elements.

Definition 1. A mapping V of $X\times B$ into $[0, +\infty]$ is called a *kernel* on X, if it has the following properties: For any $x\in X$, the set function $V(x, e)$ of e is a positive Radon measure on X, and for any relatively compact $e\in B$, the function $V(x, e)$ of x is a locally bounded Borel function.

Given a positive Borel function f, *its potential* $Vf(x)$ is defined by

$$Vf(x)=\int f(y)V(x, dy).$$

Given a positive Radon measure μ, its potential $\mu V(e)$ is defined by

$$\mu V(e)=\int V(x, e)d\mu(x)$$

provided that μV is a positive Radon measure.

We shall say that a kernel V is *continuous* if it is a positive linear mapping of C into C.[1] We consider only such a kernel in this note.

Definition 2. (*Complete maximum principle on C_K^+*) For any $f, g \in C_K^+$ and for any non-negative real number a, an inequality

$$Vf(x) \leqq Vg(x)+a$$

on S_f, *the support of* f, implies the same inequality in the whole space.

The complete maximum principle on C^+ is similarly defined.

In case $a=0$ the complete maximum principle is called *the domination principle.*

Definition 3. (*Balayage principle*) For any relatively compact open set ω in X and for any $\mu \in M_K^+$, there is a positive measure ν supported by $\bar{\omega}$, the closure of ω, such that

(1) $\nu V \leqq \mu V$ in X,

(2) the restrictions of νV and μV to ω are identical.

Replacing "any relatively compact open set" by "any open set" we have *the balayage principle on any open set.*

Given a positive linear mapping $V: C \rightarrow C$, *the balayage principle with respect to the dual mapping V^* of V* is similarly defined.

3. Balayage principle on any open set. In order to prove the theorems below we need the following

Lemma. *Assume that a continuous kernel V is strictly positive.*[2] *Then the complete maximum principle (the domination principle, resp.) on C_K^+ implies the complete maximum principle (the domination principle, resp.) on C^+.*

Proof. Suppose that for $f, g \in C^+$ and for $a \geqq 0$,

$$Vf(x) \leqq Vg(x)+a \qquad \text{on } S_f.$$

There exist a directed set $I=\{\iota\}$ and monotone increasing nets $\{f_\iota\}_{\iota \in I}$ and $\{g_\iota\}_{\iota \in I}$ of functions in C_K^+ such that $\sup_I f_\iota(x)=f(x)$ and $\sup_I g_\iota(x)=g(x)$ at any $x \in X$. Then $\{Vf_\iota\}_{\iota \in I}$ and $\{Vg_\iota\}_{\iota \in I}$ are the monotone increasing nets, and at any $x \in X$, $\sup_I Vf_\iota(x)=Vf(x)$ and $\sup_I Vg_\iota(x)=Vg(x)$. Now let ι be arbitrarily fixed in I. By the strict positivity of V there exists $h \in C_K^+$ such that $Vh(x) \geqq 1$ on S_{f_ι}. For any $\varepsilon>0$, there exists $\kappa \in I$ such that

$$Vf_\iota(x) \leqq Vg_\kappa(x)+\varepsilon Vh(x)+a \qquad \text{on } S_{f_\iota}.$$

By the complete maximum principle on C_K^+, the same inequality is valid everywhere. Hence we have

$$Vf_\iota(x) \leqq Vg(x)+a \qquad \text{everywhere,}$$

and hence

1) Deny [2] says that V is *continuous* when it transforms C_K into C.

2) It means that for any $x \in X$, there is a function f in C such that $Vf(x)>0$.

$$Vf(x) \leqq Vg(x) + a \qquad \text{everywhere.}$$

This completes the proof.

Theorem 1. *Assume that the image $V(C)$ of C by a continuous kernel V is dense in C. For V to satisfy the balayage principle on any open set, it is necessary and sufficient that V satisfies the domination principle on C_K^+.*

Proof. Necessity is evident (Deny [2]). We shall show that the domination principle on C_K^+ implies the balayage principle on any open set. To begin with we shall denote by $\tilde{X}$ the Čech compactification of X ([5], p. 276), and denote by $\tilde{f}$ the unique continuous extension of f to $\tilde{X}$. We define a positive linear mapping of $\tilde{C} = C(\tilde{X})$ into $\tilde{C}$, denoted by $\tilde{V}$, as follows:

$$\tilde{V}(\tilde{f}) = \widetilde{V(f)} \qquad \text{for any } \tilde{f} \in \tilde{C}.$$

Then $\tilde{V}(\tilde{C})$ is dense in $\tilde{C}$. Furthermore the denseness of $\tilde{V}(\tilde{C})$ in $\tilde{C}$ implies that $\tilde{V}$ is strictly positive. On the other hand, by the above lemma V satisfies the domination principle on C^+ and hence $\tilde{V}$ satisfies the domination principle on $\tilde{C}^+ (= C_K^+(\tilde{X}))$. Denoting by $\tilde{M}^+ (= M_K^+(\tilde{X}))$ the totality of positive Radon measures on $\tilde{X}$, we have

$$\langle \tilde{V}^* \tilde{\mu}, \tilde{f} \rangle = \langle \tilde{\mu}, \tilde{V}\tilde{f} \rangle \qquad \text{for any } \tilde{\mu} \in \tilde{M}^+ \text{ and for any } \tilde{f} \in \tilde{C},$$

where $\tilde{V}^* \tilde{\mu}$ is the image of $\tilde{\mu}$ by the dual mapping $\tilde{V}^*$ of $\tilde{V}$ and $\langle \lambda, g \rangle$ denotes the λ-integral of g. By Choquet-Deny's theorem [1], $\tilde{V}^*$ satisfies the balayage principle. Now let μ be a measure in M_K^+ and ω be an open set in X. Put

$$\langle \tilde{\mu}, \tilde{f} \rangle = \langle \mu, f \rangle \qquad \text{for any } \tilde{f} \in \tilde{C}.$$

Then $\tilde{\mu}$ is a measure in $\tilde{M}^+$, and there exists a balayaged measure $\tilde{\nu}$ of $\tilde{\mu}$ to the open set ω.[3] We shall show that a positive measure ν on X, defined by

$$\langle \nu, f \rangle = \langle \tilde{\nu}, \tilde{f} \rangle \qquad \text{for any } f \in C,$$

is a required balayaged measure. First let f be a function in C_K whose support is disjoint from $\bar{\omega}$. Then the support of $\tilde{f}$ is disjoint from the closure of ω in $\tilde{X}$. Hence

$$\langle \nu, f \rangle = \langle \tilde{\nu}, \tilde{f} \rangle = 0,$$

and ν is supported by $\bar{\omega}$. On the other hand, νV is a positive Radon measure and

$$\langle \nu V, f \rangle = \langle \nu, Vf \rangle \qquad \text{for any } f \in C_K.$$

Consequently for any $f \in C_K^+$ we have

$$\begin{aligned} \langle \nu V, f \rangle &= \langle \nu, Vf \rangle = \langle \tilde{\nu}, \widetilde{Vf} \rangle = \langle \tilde{\nu}, \tilde{V}\tilde{f} \rangle = \langle \tilde{V}^* \tilde{\nu}, \tilde{f} \rangle \\ &\leqq \langle \tilde{V}^* \tilde{\mu}, \tilde{f} \rangle = \langle \tilde{\mu}, \tilde{V}\tilde{f} \rangle = \langle \tilde{\mu}, \widetilde{Vf} \rangle = \langle \mu, Vf \rangle = \langle \mu V, f \rangle. \end{aligned}$$

Hence $\nu V \leqq \mu V$. Similarly we obtain

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3) Being open in X, ω is open in $\tilde{X}$.

$$\langle \nu V, f\rangle = \langle \mu V, f\rangle,$$

for any $f \in C_K$ such that S_f is contained in ω, and hence the restrictions of νV and μV to ω are identical. This completes the proof.

Remark 1. It is an open question if the condition, the denseness of $V(C)$ in C, can be replaced by a weaker condition.

Remark 2. Let Ω be a bounded domain of the $n(\geqq 2)$-dimensional Euclidean space and $G(x, y)$ be the Green function of Ω. Then the *Green kernel* $V(x, e)=\int_e G(x, y)dy$ satisfies the conditions of Theorem 1, and hence it satisfies the balayage principle on any open set.

4. Continuous kernel of Hunt. In this section we prove the following

Theorem 2. *If a continuous kernel V has the two properties*:

(α) *$V(C)$ is dense in C,*

(β) *V satisfies the complete maximum principle on C_K^+,*

then there exists uniquely a family $\{P_t\}_{t\geqq 0}$ of positive linear operators of C into C such that

(i) *$\{P_t\}_{t\geqq 0}$ is a semi-group,*

(ii) *$\|P_t\|\leqq 1$ for any $t\geqq 0$,*

(iii) *$\|P_tf-f\|\to 0$ as $t\to +0$, for any $f\in C$,*

(iv) *$Vf=\int_0^\infty P_tfdt$ for any $f\in C$.*[4]

Proof. Similarly as in the proof of Theorem 1, we have

($\tilde{\alpha}$) $\tilde{V}(\tilde{C})$ is dense in $\tilde{C}$,

($\tilde{\beta}$) $\tilde{V}$ satisfies the complete maximum principle on $\tilde{C}^+$.

Hence by Hunt's theorem (Lion [4]), there is a family $\{\tilde{P}_t\}_{t\geqq 0}$ of positive linear operators of $\tilde{C}$ into $\tilde{C}$ such that

($\tilde{\text{i}}$) $\{\tilde{P}_t\}_{t\geqq 0}$ is a semi-group,

($\tilde{\text{ii}}$) $\|\tilde{P}_t\|\leqq 1$ for any $t\geqq 0$,

($\tilde{\text{iii}}$) $\|\tilde{P}_t\tilde{f}-\tilde{f}\|\to 0$ as $t\to +0$, for any $\tilde{f}\in \tilde{C}$,

($\tilde{\text{iv}}$) $\tilde{V}\tilde{f}=\int_0^\infty \tilde{P}_t\tilde{f}dt$ for any $\tilde{f}\in\tilde{C}$.

For any $f\in C$ and for any $t\geqq 0$, put

$$P_tf = \text{the restriction of } \tilde{P}_t\tilde{f} \text{ to } X.$$

Then we can see immediately that $\{P_t\}_{t\geqq 0}$ satisfies (i), (ii), (iii), and (iv). For the uniqueness of the semi-group we refer to Deny [2]. This completes the proof.

Remark 3. Hunt's theorem is stated for the continuous kernels which transform C_K into C_0, where C_0 denotes the totality of continuous functions on X vanishing at infinity. Consequently his theorem says nothing about the Green kernel (defined in Remark 2)

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4) If there is such a semi-group, V is called a *continuous kernel of Hunt* (Deny [3]).

on Ω whose boundary has irregular points, since $Vf \notin C_0$ even if $f \in C_K$. But it is a continuous kernel of Hunt by Theorem 2.

References

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