99. *A Generalized Derivative and Integrals of the Perron Type*

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1. Introduction. Many kinds of integration of the Perron type have been given by various authors using various types of generalized derivatives.

The aim of this paper is to introduce axiomatically a generalized derivative which includes ordinary derivative, approximate derivative and Cesàro derivative, and to build an integral of the Perron type including ordinary Perron integral, AP- and CP-integral defined by J. C. Burkill [1], [2] and more generally G. Sunouchi and M. Utagawa's generalized Perron integral [4], [3].

2. A generalized derivative. Definition 2.1. Let M be a linear space of measurable functions defined on closed interval $[a, b]$. If we can assign uniquely the extented real value $\underline{GD}f(x)$ to any function $f(x)\in M$ and any point $x\in[a, b]$ such that

(i) $\underline{GD}1=0$,

(ii) $\underline{GD}[f(x)+g(x)]\geqq\underline{GD}f(x)+\underline{GD}g(x)$,

(iii) if $f(x)$ is ordinary differentiable at x then

$$\underline{GD}[f(x)+g(x)]=Df(x)+\underline{GD}g(x),$$

(iv) $\underline{GD}f(x)\geqq\underline{D}f(x)$

where $\underline{D}f(x)$ means ordinary lower derivate of f at x.

(v) $\underline{GD}[\alpha f(x)]=\alpha\,\underline{GD}f(x)\qquad(\alpha>0)$,

then $\underline{GD}f(x)$ is termed generalized lower derivate of $f(x)$ at x.

Throughout this paper we more assume the following property.

(vi) If $f\in M$ and $\underline{GD}f(x)\geqq 0$ at each point x of $[a, b]$ then $f(x)$ is non-decreasing.

Definition 2.2. If we define $\overline{GD}f(x)$ by $\overline{GD}f(x)=-\underline{GD}[-f(x)]$ then $\overline{GD}f(x)$ is called generalized upper derivate of $f(x)$ at x. If $\underline{GD}f(x)=\overline{GD}f(x)$ then we say that $f(x)$ has generalized derivative at x and the common value is written by $GDf(x)$.

Ordinary-, approximate- and Cesàro-lower derivate satisfy the conditions (i)-(vi). The proofs of (vi) for approximate- and Cesàro-derivate were given by G. Sunouchi and M. Utagawa [4].

We can easily prove the following properties.

(1) $\overline{GD}1=0$.

(2) $\overline{GD}[f(x)+g(x)]\leqq\overline{GD}f(x)+\overline{GD}g(x)$.

(3) If $f(x)$ is ordinary differentiable at x then

$$\overline{GD}[f(x)+g(x)]=Df(x)+\overline{GD}\,g(x).$$

(4) $\overline{GD}f(x)\leqq \bar{D}f(x)$

where $\bar{D}f(x)$ means ordinary upper derivate of $f(x)$.

(5) $\overline{GD}[\alpha f(x)]=\alpha\,\underline{GD}f(x)$ and $\underline{GD}[\alpha f(x)]=\alpha\,\overline{GD}f(x)$ for $\alpha<0$.

$\overline{GD}[\alpha f(x)]=\alpha\,\overline{GD}f(x)$ for $\alpha>0$.

(6) If C is constant then $GD\ C=0$.

(7) $\underline{GD}f(x)\leqq\overline{GD}f(x)$.

3. Generalized Perron integral. Definition 3.1. A function $U(x)\in M$ is termed upper function of $f(x)$ in $[a, b]$ if

(i) $U(a)=0$,

(ii) $\underline{GD}\,U(x)>-\infty$ at each point x,

(iii) $\underline{GD}\,U(x)\geqq f(x)$ at each point x.

A lower function $L(x)$ is defined correspondingly.

Definition 3.2. If $f(x)$ has upper and lower functions in $[a, b]$ and $\inf_U U(b)=\sup_L L(b)$ finite then $f(x)$ is termed Perron integrable in the generalized sense. The common value of the two bounds is called the definite GP-integral of $f(x)$ and is denoted by $(GP)\int_a^b f(t)dt$.

Theorem 3.1. For any upper and lower functions $U(x)$, $L(x)$ of $f(x)$ in $[a, b]$, $\omega(x)=U(x)-L(x)$ is non-decreasing on $[a, b]$.

Proof. By (ii) of Definition 2.1 and Definition 2.2,

$$\underline{GD}\,\omega(x)\geqq\underline{GD}\,U(x)+\underline{GD}[-L(x)]$$
$$=\underline{GD}\,U(x)-\overline{GD}\,L(x).$$

It follows from (ii) and (iii) in Definition 3.1 that $\underline{GD}\,\omega(x)\geqq 0$ at each point x. Hence by axiom (vi), $\omega(x)$ is non-decreasing on $[a, b]$.

Since $U(b)\geqq L(b)$ by Theorem 3.1, it follows from Definition 3.2 that a necessary and sufficient condition that $f(x)$ is GP-integrable over $[a, b]$ is that for a given $\varepsilon>0$ there exists an upper function $U(x)$ and a lower function $L(x)$ such that $U(b)<L(b)+\varepsilon$.

Theorem 3.2. If $f(x)$ is GP-integrable on $[a, b]$ then it is also GP-integrable on every sub-interval $[c, d]$.

Proof. Since $f(x)$ is GP-integrable on $[a, b]$, for a given $\varepsilon>0$ there exist upper and lower functions $U(x)$, $L(x)$ such that $0\leqq U(b)-L(b)<\varepsilon$. If we define the functions $U*(x)$ and $L*(x)$ on $[c, d]$ as $U*(x)=U(x)-U(c)$, $[L*(x)=L(x)-L(c)]$ then $U*(c)=L*(c)=0$, $\underline{GD}\,U*(x)=\underline{GD}U(x)>-\infty$ and $\underline{GD}\,U*(x)\geqq f(x)[\overline{GD}\,L*(x)<+\infty$, $\overline{GD}L*(x)\leqq f(x)]$ by property (6) and axiom (iv) [by properties (6) and (3)]. Hence $U*(x)[L*(x)]$ is an upper [lower] function of $f(x)$ on $[c, d]$ and $U*(d)-L*(d)\leqq U(d)-L(d)<\varepsilon$. This completes the proof.

Theorem 3.3. If $f(x)$ is GP-integrable on $[a, b]$ then for $a<c<b$

$$(GP)\int_a^b f(t)dt=(GP)\int_a^c f(t)dt+(GP)\int_c^b f(t)dt.$$

Proof. By Theorem 3.2 $f(x)$ is GP-integrable on $[a,c]$ and $[c,b]$ respectively. Let $U(x)$ be any upper function of $f(x)$ in $[a,b]$. Then $U(x)$ is an upper function of $f(x)$ on $[a,c]$, and $U(x)-U(c)$ is an upper function of $f(x)$ on $[c,d]$. Since $U(b)=U(c)+[U(b)-U(c)]$, we have

$$(GP)\int_a^b f(t)dt\geqq(GP)\int_a^c f(t)dt+(GP)\int_c^b f(t)dt.$$

Similarly using lower functions we obtain

$$(GP)\int_a^b f(t)dt\leqq(GP)\int_a^c f(t)dt+(GP)\int_c^b f(t)dt,$$

which proves the equality.

Theorem 3.4. If f and g are GP-integrable on $[a,b]$ then $\alpha f+\beta g$ is also GP-integrable on $[a,b]$ and

$$(GP)\int_a^b[\alpha f(t)+\beta g(t)]dt=\alpha(GP)\int_a^b f(t)dt+\beta(GP)\int_a^b g(t)dt.$$

Proof. Since $f(x)[g(x)]$ is GP-integrable there exist $U_1(x)$ and $L_1(x)[U_2(x)$ and $L_2(x)]$ such that

$$0\leqq U_1(b)-L_1(b)<\varepsilon \qquad [0\leqq U_2(b)-L_2(b)<\varepsilon].$$

(i) First we prove that αf is GP-integrable and

$$(GP)\int_a^b \alpha f(t)dt=\alpha(GP)\int_a^b f(t)dt.$$

For the case $\alpha=0$ it is clear. If $\alpha>0$ then we put

$$U(x)=\alpha U_1(x) \qquad [L(x)=\alpha L_1(x)].$$

It follws from axiom (v) and property (5) that

$$\underline{GD}\,U(x)\geqq\alpha f(x) \text{ and } \underline{GD}\,U(x)>-\infty$$

$$[\overline{GD}\,L(x)\leqq\alpha f(x) \text{ and } \overline{GD}\,L(x)<+\infty].$$

Hence $U[L]$ is an upper [lower] function of f in $[a,b]$ and

$$0\leqq U(b)-U(a)=\alpha(U_1(b)-L_1(b))<\alpha\varepsilon$$

which proves the integrability of αf. Since

$$L_1(b)\leqq(GP)\int_a^b f(t)dt\leqq U_1(b)$$

and $\alpha>0$, we have

$$L(b)\leqq\alpha(GP)\int_a^b f(t)dt\leqq U(b).$$

On the other hand

$$L(b)\leqq(GP)\int_a^b \alpha f(t)dt\leqq U(b),$$

and therefore

$$(GP)\int_a^b \alpha f(t)dt=\alpha(GP)\int_a^b f(t)dt.$$

For the case $\alpha<0$ we can prove the above equality similarly.

(ii) Next we shall show that if f and g are GP-integrable then $f+g$

is also so and

$$(GP)\int_a^b [f(t)+g(t)]\,dt=(GP)\int_a^b f(t)dt+(GP)\int_a^b g(t)dt.$$

Let $U(x)=U_1(x)+U_2(x)$ and $L(x)=L_1(x)+L_2(x)$. Then $U(x)[L(x)]$ is an upper [lower] function of $f+g$ by axiom (ii) and property (2). Moreover $U(b)-L(b)<2\varepsilon$. Hence $f+g$ is GP-integrable and

$$(GP)\int_a^b f(t)dt+(GP)\int_a^b g(t)dt=(GP)\int_a^b [f(t)+g(t)]dt.$$

The general case can be deduced from (i) and (ii).

Definition 3.3. Let $f(x)$ be a function GP-integrable on $[a, b]$. We define the indefinite GP-integral of $f(x)$ as

$$F(x)=(GP)\int_a^x f(t)dt.$$

Theorem 3.5. If $f(x)$ is GP-integrable on $[a, b]$ then for any upper [lower] function $U(x)$ $[L(x)]$ the function $U(x)-F(x)$ $[F(x)-L(x)]$ is non-decreasing on $[a, b]$.

Proof. Let $a\leqq x_1<x_2\leqq b$. Then $U*(x)=U(x)-U(x_1)$ is an upper function of $f(x)$ in $[x_1, x_2]$. Hence

$$U(x_2)-U(x_1)\geqq(GP)\int_{x_1}^{x_2} f(t)dt,$$

and by Theorem 3.3,

$$U(x_2)-U(x_1)\geqq F(x_2)-F(x_1).$$

Similarly we can prove the theorem for the function $F(x)-L(x)$.

Theorem 3.6. The function $F(x)$ is differentiable in the generalized sense at almost all points of $[a, b]$ and

$$GD\,F(x)=f(x) \qquad \text{a.e.}$$

Proof. For a given $\varepsilon>0$, we can find an upper function $U(x)$ such that $U(x)-F(x)<\varepsilon^2$. If we put $R(x)=U(x)-F(x)$ then $R(x)$ is non-decreasing, and therefore it has finite ordinary derivative $R'(x)$ almost everywhere which is summable on $[a, b]$. Hence

$$(L)\int_a^b R'(x)dx\leqq R(b)-R(a)=U(b)-F(b)<\varepsilon^2.$$

We set $A(\varepsilon)=\{x: \underline{GD}\,F(x)<f(x)-\varepsilon\}$, and denote by S the set of points where $R'(x)$ exists and finite. Then we have $|S|=b-a$. If $x\in A(\varepsilon)$ then

$$\underline{GD}\,F(x)<f(x)-\varepsilon\leqq\underline{GD}\,U(x)-\varepsilon,$$

and therefore

(1) $$\underline{GD}\,U(x)-\underline{GD}\,F(x)>\varepsilon.$$

For any point $x\in S$, by axiom (iii),

(2) $$R'(x)=\underline{GD}\,U(x)-\underline{GD}\,F(x),$$

and if we put $B(\varepsilon)=\{x: R'(x)>\varepsilon\}$ then it follows from (1) and (2) that $x\in A(\varepsilon)\cdot S$ implies $x\in B(\varepsilon)$, i.e. $A(\varepsilon)\cdot S\subset B(\varepsilon)$. Hence

(3) $$|A(\varepsilon)|\leqq|B(\varepsilon)|.$$

But

$$\varepsilon\,|B(\varepsilon)|\leqq(L)\int_{B(\varepsilon)}R'(t)dt\leqq(L)\int_a^b R'(t)dt<\varepsilon^2.$$

Hence $|B(\varepsilon)|<\varepsilon$, and by (3), $|A(\varepsilon)|<\varepsilon$. Since

$$\{x: \underline{GD}\,F(x)<f(x)\}=\bigcup_{k=1}^{\infty}\left\{x: \underline{GD}F(x)<f(x)-\frac{\varepsilon}{2^k}\right\},$$

we have

$$|\{x: \underline{GD}\,F(x)<f(x)\}|\leqq\sum_{k=1}^{\infty}\left|A\left(\frac{\varepsilon}{2^k}\right)\right|<\sum_{k=1}^{\infty}\frac{\varepsilon}{2^k}=\varepsilon.$$

Consequently we obtain $\underline{GD}\,F(x)\geqq f(x)$ a.e.

Using a lower function, it can be proved analogously that

$$\overline{GD}\,F(x)\leqq f(x) \qquad \text{a.e.}$$

Theorem 3.7. If $f(x)$ is ordinary Perron-integrable (P-integrable) on $[a, b]$ then it is GP-integrable on $[a, b]$ and

$$(P)\int_a^b f(t)dt=(GP)\int_a^b f(t)dt.$$

Proof. The upper [lower] function $M(x)$ [$m(x)$] of defining the ordinary Perron integral of $f(x)$ over $[a, b]$ has the following properties; $M(a)=m(a)=0$, $\underline{D}\,M(x)>-\infty$ and $\underline{D}\,M(x)\geqq f(x)$ at each point of $[a, b]$ [$\bar{D}\,m(x)<+\infty$, $\bar{D}\,m(x)\leqq f(x)$ at each point]. Hence, by axiom (iv) and property (4), any $M(x)$ [$m(x)$] is an upper [lower] function of $f(x)$ in the GP-sense. From the inequalities

$$\inf U(b)\leqq\inf M(b)=\sup m(b)\leqq\sup L(b),$$

and $U(b)\geqq L(b)$, we have $\inf U(b)=\sup L(b)=(P)\int_a^b f(t)dt$.

Theorem 3.8. A non-negative function $f(x)$ which is GP-integrable on $[a, b]$ is necessarily L-integrable on $[a, b]$ and both integrals over $[a, b]$ coincide each other.

Proof. Let $U(x)$ be any upper function of $f(x)$. Since $\underline{GD}\,U(x)\geqq f(x)\geqq 0$ it follows from axiom (vi) that $U(x)$ is non-decreasing. Hence $U(x)$ is ordinary differentiable at almost all points, and therefore $U'(x)$ is summable. It follows from axiom (iv), properties (4) and (7) that $U'(x)=GD\,U(x)$ a.e. Hence $U'(x)\geqq f(x)\geqq 0$ a.e. which implies L-integrability of $f(x)$ on $[a, b]$. The remain part of the theorem follows from Theorem 3.7.

Theorem 3.9. Given a non-decreasing sequence f_n of functions which are GP-integrable on $[a, b]$ and whose GP-integral over $[a, b]$ constitute a sequence bounded above, the function $f(x)=\lim\limits_{n\to\infty} f_n(x)$ is itself GP-integrable on $[a, b]$ and we have

$$(1) \qquad (GP)\int_a^b f(t)dt=\lim\,(GP)\int_a^b f_n(t)dt.$$

Proof. Since f_n-f_1 is non-negative, it follows from Theorem

3.8 that f_n-f_1 is L-integrable. Hence, by Lebesgue's theorem, we have

$$\lim_{n\to\infty}(L)\int_a^b(f_n-f_1)dt=(L)\int_a^b(f-f_1)dt. \tag{2}$$

Since the sequence of integrals $(GP)\int_a^b f_n(t)dt$ is bounded above, the sequence of integrals

$$(L)\int_a^b(f_n-f_1)dt=(GP)\int_a^b(f_n-f_1)dt$$

is also so, and therefore

$$0\leqq(L)\int_a^b(f-f_1)dt<\infty$$

which implies L-integrability of the function $f-f_1$. Hence $f-f_1$ is GP-integrable, and f is also so. The equality (1) follows directly from (2) and Theorem 3.4.

If we put the approximate derivates $\underline{AD}$ and $\overline{AD}$ in place of the generalized derivates $\underline{GD}$ and $\overline{GD}$ respectively in the Definition 3.1 and M is the set of all measurable functions defined on $[a, b]$ then we have the approximately continuous Perron integral defined by G. Sunouchi and M. Utagawa [4] which is more general than Burkill's AP-integral [1]. Also if, in the Definition 3.1, M is the set of all special Denjoy integrable functions on $[a,b]$ and the generalized derivates are replaced by the Cesàro derivates then Definition 3.2 defines Sunouchi and Utagawa's Cesàro-Perron integral [4] which is known to be equivalent to Burkill's Cesàro-Perron integral [3].

References

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