125. On Axiom Systems of Propositional Calculi. IV

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Recently, in his book [5], E. Mendelson gave an axiom system for two valued propositional calculus. His axiom system is written by Lukasiewicz symbols as follows:

1 *CpCqp*,

2 *CCpCqrCCpqCpr*,

3 *CCNpNqCCNpqp*.

E. Mendelson [5] proved some tautologies by using the rules of inference and a metatheorem known as Herbrand deduction theorem: If Γ is a set of theses and p, q are theses and $\Gamma, p \vdash q$, then $\Gamma \vdash p \supset q$ (see J. Herbrand [2] or A. A. Mullin [6]).

In this note, we shall use only rules of substitution and detachment, and prove some theses.

The first two axioms 1 and 2 are theses 18 and 35 in J. Lukasiewicz [4] respectively. The axiom 3 is also a thesis in Lukasiewicz (L_1)-system (see Y. Imai and Iséki [3]). It follows from 49: *CCNpNqCqp* and 15: *CCNpqCCqpp* in [4]. To prove it, we shall use the following two fundamental theses:

a) *CCqrCCpqCpr*,

b) *CCpCqrCqCpr*.

These tautologies are theses 22 and 21 in [4] respectively.

b) *p/CNpq, q/Cqp, r/p* **C*15 *p/q, q/p*—4,

4 *CCpqCCNqpq*.

a) *p/CNqNp, q/Cpq, r/CCNqpq* **C*4—*C*49 *p/q, q/p*—3,

3 *CCNqNpCCNqpq*.

This shows that axiom 2 is a thesis in L_1-system. In the third note [1], Y. Arai has proved that *CpCqp*, *CCpCqrCCpqCpr* imply the following important theses:

1′ *CCpqCCqrCpr*,

2′ *CCqrCCpqCpr*,

and

3′ *CCpCqrCqCpr*.

We shall now proceed to prove the Lukasiewicz (L_1)-axioms 1: *CCpqCCqrCpr*, 2: *CCNppp*, and 3: *CpCNpq*.

From remarks given above, we have *CCpqCCqrCpr*. We shall show that axioms 1, 2, and 3 imply *CCNppp*. The proof is done by the following lines.

2 r/p *C1—5,

5 CCpqCpp.

5 q/Cqp *C1—6,

6 Cpp.

3 q/p *C6 p/Np—7,

7 CCNppp.

The thesis CpCNpq follows from the following process.

1 p/Np, q/Nq—8,

8 CNpCNqNp.

1′ p/Np, q/CNqNp, r/CCNqpq *C8—C3—9,

9 CNpCCNqpq.

3′ p/Np, q/CNqp, r/q *C9—10,

10 CCNqpCNpq.

2′ q/CNqp, r/CNpq *C10—C1 q/Nq—11,

11 CpCNpq.

Therefore we have proved that axioms 1, 2, and 3 imply axioms of (L_1)-system. We shall further prove some theses.

10 p/Np, q/p *C6—12,

12 CNNpp.

1′ p/NNp, q/p, r/q *C12—13,

13 CCpqCNNpq.

1′ p/Cpq, q/CNNpq, r/CNqNp *C13—C10 p/q, q/Np—14,

14 CCpqCNqNp.

2′ q/NNq, r/q *C12—15,

15 CCpNNqCpq.

3′ q/Np, r/q *C11—16,

16 CNpCpq.

1′ p/Np, q/Cpq *C16—17,

17 CCCpqrCNpr.

1′ p/Cpq, q/CNqNp, r/CCNqpq *C14—C3—18,

18 CCpqCCNqpq.

3′ p/Cpq, q/CNqp, r/q *C18—19,

19 CCNqpCCpqq.

19 q/Np *C12—20,

20 CCpNpNp.

1′ q/CNpq *C11—21,

21 CCCNpqrCpr.

21 q/NNp, r/NNp *C20 p/Np—22,

22 CpNNp.

1′ q/NNp, r/q *C22—23,

23 CCNNpqCpq.

$2'$ r/NNq, $*C22$ p/q—24,

24 $CCpqCpNNq$.

$1'$ $p/CNqNp$, $q/CNNpq$, r/Cpq $*C10$ p/Np—$C23$—25,

25 $CCNqNpCpq$.

$1'$ $p/CpNq$, $q/CNNqNp$, $r/CqNp$ $*C14$ q/Nq—$C23$ p/q, q/Np—26,

26 $CCpNqCqNp$.

$1'$ $p/CNpq$, $q/CNpq$, $r/CCpqq$ $*C10$ p/q, q/p—$C19$—27,

27 $CCNpqCCpqq$.

$3'$ $p/CNpq$, q/Cpq, r/q $*C27$—28,

28 $CCpqCCNpqq$.

From the theses given above, we have axioms systems of Frege, Russell, Hilbert, and Lukasiewicz (L_3) (see [3]). By thesis 19, we see that our axioms imply another system of Lukasiewicz discussed in our second note by Y. Arai.

References

[1] Y. Arai: On axiom systems of propositional calculi. III. Proc. Japan Acad., **41**, 570-574 (1965).

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