123. Non-negative Integer Valued Functions on Commutative Groups. I

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(Comm. by Kinjirô KUNUGI, M.J.A., Sept. 13, 1965)

T. Tamura, one of the authors, introduced "an indexed group" which means a commutative group G with a non-negative integer valued function $I(x, y)$ defined on $G\times G$ and satisfying the following conditions:

(A) $I(x, y)=I(y, x)$

(B) $I(x, y)+I(xy, z)=I(x, yz)+I(y, z)$ for any $x, y, z\in G$

(C) For any $x\in G$, there is a positive integer m (depending on x) such that $I(x^m, x)>0$.

(D) $I(e, e)=1$ where e is the identity of G.

It was shown in [1] that $I(e, x)=1$ for all $x\in G$ for every indexed group G. Consequently if G is periodic, condition (C) is satisfied whenever conditions (A), (B), and (D) are satisfied.

Given an indexed group G, there is a commutative archimedean cancellative semigroup without idempotent such that the fundamental group of which is isomorphic to the group G (Theorem 4 in [1] or Exercise § 4.3, 8. p. 136 in [2]).

The purpose of this paper, as one of the series, is to show how all I-functions on a finitely generated commutative group G may be obtained.

1. The Case where G is a Finite Cyclic Group. Suppose G is a cyclic group of order n generated by a. Let $E(i, j, k)$ denote the equation obtained by setting x, y, z as a^i, a^j, a^k respectively in (B), and let $E'(i, j, k)$ be the equation obtained by exchanging the two sides of $E(i, j, k)$ with each other.

Lemma 1. *$E(m, p, q)$, $m>0$, p, q integers, is expressed by equations of type $E(1, p, q)$.*

Proof. If $m=1$, it is obvious. Let $m\geqq 2$, then $E(m, p, q)$ is obtained by adding $E(m-1, 1, p)$, $E'(m-1, 1, p+q)$, $E(m-1, p+1, q)$ and $E(1, p, q)$. By induction we get this lemma.

For integers $i(\geqq 0)$, m, n we define

$$[m, n]_i=\sum_{k=0}^{i-1} I(a, a^{m+k})-\sum_{k=0}^{i-1} I(a, a^{n-k}), \qquad \text{if} \quad i>0$$
$$=0 \qquad\qquad \text{if} \quad i=0$$

Adding then, $E(1, 1, j), E(1, 2, j), \cdots, E(1, i-1, j)$, we obtain

Lemm 2.

$$I(a^i, a^j)=I(a, a^{i+j-1})+[j, i-1]_{i-1} \quad \text{for} \quad i\geqq 1.$$

Conversely if $I(a, a^k)$, for all k, are given and if $I(a^i, a^j)$ is defined in this manner, we can easily prove that the function I satisfies (B).

Theorem 1. *If G is a cyclic group of order n, the function values $I(a, a^k)$, $k=1, \cdots, n-1$, are independent up to relative size considerations and every other function value can be determined from these $n-1$ values by the form in Lemma 2.*

Next we shall consider determining the relative sizes of the "independent" elements $I(a, a^k)$, $k=1, \cdots, n-1$. The major conditions are $I(a^i, a^j)\geqq 0$ for all $i, j=1, \cdots, n-1$. We note that

$$0\leqq I(a, a)\leqq I(a, a^2)\leqq \cdots \leqq I(a, a^{n-1})$$

is sufficient for a solution. In fact, in this case, it follows that for $2\leqq i\leqq j\leqq n-1$.

(1.1) if $i+j-1\leqq n$, then

$$I(a^i, a^j)=I(a, a^{i+j-1})+\sum_{k=0}^{i-2}(I(a, a^{j+k})-I(a, a^{1+k}))\geqq 0$$

since $n>j+k>1+k$ for all k with $0\leqq k\leqq i-2$.

(1.2) if $i+j-1>n$, then we can put $i+j-1=n+s$, $1\leqq s\leqq n-3$ and

$$\begin{aligned} I(a^i, a^j)&=\left(\sum_{k=j}^{n} I(a, a^k)+\sum_{k=n+1}^{i+j-1} I(a, a^k)\right)-\left(\sum_{k=1}^{s} I(a, a^k)+\sum_{k=s+1}^{i-1} I(a, a^k)\right)\\ &=I(a, a^n)+\sum_{k=0}^{n-j-1}(I(a, a^{j+k})-I(a, a^{s+1+k}))\geqq 1 \end{aligned}$$

since $n>j+k>s+1+k$ for all k with $0\leqq k\leqq n-j-1$.

If $n\leqq 4$, then the following conditions for $I(a, a^k)$, $k=1, \cdots, n-1$, are obtained easily:

(2.1) the case $n=2$, $I(a, a)\geqq 0$

(2.2) the case $n=3$, $I(a, a)\geqq 0$, $I(a, a^2)\geqq \max\{0, I(a, a)-1\}$.

(2.3) the case $n=4$, $I(a, a)\geqq 0$, $I(a, a^2)\geqq 0$

$$I(a, a^3)\geqq \max\{0, I(a, a)-I(a, a^2), I(a, a)-1, I(a, a^2)-1\}.$$

So, hereafter, we assume $n\geqq 5$. By (A) we may consider the conditions for $I(a, a^k)$, $k=1, \cdots, n-1$ under $I(a^i, a^j)\geqq 0$ for all i, j such that $2\leqq i\leqq j\leqq n-1$. From Lemma 2 and $I(a^i, a^j)\geqq 0$, we get an inequality

(3) $I(a, a^{i+j-1})\geqq [1, i+j-2]_{i-1}$.

Putting here $i+j-1=k$, k runs through $3, 4, \cdots, 2n-3$ and for a fixed k such that $3\leqq k\leqq 2n-3$ all the inequalities (3) are given as follows:

(4.1) the case $3\leqq k\leqq n-1$

$$I(a, a^k)\geqq [1, k-1]_{i-1},\ i=2, 3, \cdots, \left[\frac{k+1}{2}\right]$$

(4.2) the case $k=n$

$$I(a, a^n) \geqq [1, n-1]_{i-1}, \quad i=2, 3, \cdots, \left[\frac{n+1}{2}\right]$$

hence

$$I(a, a^{n-1}) \geqq [1, n-2]_{i-2} + I(a, a^{i-1}) - 1$$

for all i with $2 \leqq i \leqq \left[\frac{n+1}{2}\right]$

hence

(4.3) the case $k=n+s$, $1 \leqq s \leqq n-5$, $n>5$

$$I(a, a^{n+s}) \geqq [1, n+s-1]_{i-1}, \quad i=s+2, s+3, \cdots, \left[\frac{n+s+1}{2}\right]$$

hence

$$I(a, a^{n-1}) \geqq [s+1, n-2]_{i-s-2} + I(a, a^{i-1}) - 1$$

for all i with $s+2 \leqq i \leqq \left[\frac{n+s+1}{2}\right]$.

(4.4) the case $k=2n-4$

$$I(a, a^{2n-4}) \geqq [1, 2n-5]_{i-1}, \quad i=n-2,$$

hence

$$I(a, a^{n-1}) \geqq I(a, a^{n-3}) - 1.$$

(4.5) the case $k=2n-3$

$$I(a, a^{2n-3}) \geqq [1, 2n-4]_{i-1}, \quad i=n-1,$$

hence

$$I(a, a^{n-1}) \geqq I(a, a^{n-2}) - 1.$$

Summarizing the above inequalities and $I(a, a^k) \geqq 0$, we have the following theorem.

Theorem 2. *Let G be a cyclic group of order n. $I(a^i, a^j) \geqq 0$ for all non-zero integers i, j if and only if $I(a, a^k)$, $k=1, \cdots, n-1$, satisfy the following conditions:*

(5.1) *In the cases $n=2, 3, 4$, (2.1), (2.2), (2.3) hold respectively.*

(5.2) *In the case $n \geqq 5$,*

$$I(a, a^k) \geqq \bar{m}(k) \quad k=1, 2, \cdots, n-2$$

$$I(a, a^{n-1}) \geqq \max\left\{\bar{m}(n-1), \bar{m}'(0), \bar{m}'(1), \cdots, \bar{m}'(n-5) \max_{1 \leqq i \leqq n-2} \{I(a, a^i)-1\}\right\}$$

where

$$\bar{m}(k) = \max_{0 \leqq i \leqq \left[\frac{k-1}{2}\right]} \{[1, k-1]_i\},$$

and

$$\bar{m}'(k) = \max_{1 \leqq i \leqq \left[\frac{n-k-3}{2}\right]} \{[k+1, n-2]_i + I(a, a^{i+k+1}) - 1\}.$$

We notice that the types of $I(a, a^s)$ which appear in $\bar{m}(k)$ are all $s<k$, and the types of $I(a, a^s)$ in

$$\max\left\{\bar{m}(n-1), \bar{m}'(0), \cdots, \bar{m}'(n-5), \max_{1 \leqq i \leqq n-2} \{I(a, a^i)-1\}\right\}$$

are all $s<n-1$.

2. The Case where G is an Infinite Cyclic Group. Let G be an infinite cyclic group generated by a:

$G=\{a^m;\, m=0, \pm 1, \pm 2, \cdots\}$ where a^0 is the identity element of G.

Lemma 3. *$E(m, p, q)$, m, p, q integers, is expressed by equations of type $E(1, p, q)$.*

Proof. If $m\geqq 1$, the lemma is true by Lemma 1. If $m=0$, $E(0, p, q)$ reduces to an identity. $E(-1, p, q)$ is obtained by adding $E'(1, -1, p)$, $E(1, -1, p+q)$, and $E'(1, p-1, q)$. For $m'\geqq 2$, $E(-m', p, q)$ is obtained by adding $E(-m'+1, -1, p)$, $E'(-m'+1, -1, p+q)$, $E(-m'+1, p-1, q)$, and $E(-1, p, q)$. The lemma follows by induction.

Lemma 4. *For any integer j, it holds that*

(6.1) $I(a^i, a^j)=I(a, a^{i+j-1})+[j, i-1]_{i-1}$ *if* $i\geqq 2$

(6.2) $I(a^i, a^j)=I(a, a^i)+[i+1, j-1]_{-1}$ *if* $i\leqq -1$.

Proof. The former is shown in the same way as Lemma 2, the latter is proved by adding $E(1, i, j), E(1, i+1, j), \cdots, E(1, -1, j)$.

From Lemmas 3 and 4 we have:

Theorem 3. *If G is an infinite cyclic group, the function values $I(a, a^k)$, $k=\pm 1, \pm 2, \cdots$, are independent up to relative size considerations and every function value is determined from these $I(a, a^k)$, $k=\pm 1, \pm 2, \cdots$.*

Moreover we have,

Theorem 4. *Let G be an infinite cyclic group. $I(a^i, a^j)\geqq 0$ for all non-zero integers i, j and they determine an I-function if and only if* (7.1), (7.2), *and* (7.3) *below are satisfied:*

Let

$$\bar{m}(k)=\max_{0\leqq i\leqq\left[\frac{k-1}{2}\right]}\{[1, k-1]_i\},\quad \bar{n}(-k)=\max_{0\leqq i}\{[1, i-k]_i\},$$

$$\bar{n}'(-k)=\min_{0\leqq i\leqq\left[\frac{k-2}{2}\right]}\{I(a, a^{-1})+1+[-i-1, i-k]_i\}.$$

(7.1) $I(a, a^k)\geqq \bar{m}(k)$, $k=1, 2, \cdots$

(7.2) $\begin{cases} I(a, a^{-1})\geqq \bar{n}(-1) \\ \bar{n}'(-k)\geqq I(a, a^{-k})\geqq \bar{n}(-k), \quad k=1, 2, \cdots \end{cases}$

(7.3) *For any integer $s(\neq 0)$ there exists a positive integer t_s such that*

$$[st_s, s-1]_s\geqq 0 \quad \textit{if} \quad s\geqq 1$$
$$[s, st_s-1]_{-s}\geqq 0 \quad \textit{if} \quad s\leqq -1.$$

Proof. Suppose $I(a^i, a^j)\geqq 0$ for all non-zero integers i, j. By (A) it suffices to consider $I(a^i, a^j)\geqq 0$ in the following cases:

(i) $2\leqq i\leqq j$, (ii) $2\leqq i$ and $j\leqq -1$, (iii) $j\leqq i\leqq -1$.

In each case, from Lemma 4 and $I(a^i, a^j)\geqq 0$, we get inequalities

(8.1) $I(a, a^{i+j-1})\leqq [1, i+j-2]_{i-1}$ in (i)

(8.2) $I(a, a^j)\geqq [i+j, 0]_{-j}$ in (ii)

(8.3) $I(a, a^{i+j}) \leqq I(a, a^{-1})+1+[i, j-1]_{-i-1}$ in (iii).

If we let $k=i+j-1$ in (8.1),

$$I(a, a^k) \geqq [1, k-1]_{i-1} \quad i=2, \cdots, \left[\frac{k+1}{2}\right];$$

if $k=-j$ in (8.2),

$$I(a, a^{-k}) \geqq [i-k, 0]_k \quad i=2, 3, \cdots;$$

if $k=-i-j$ in (8.3),

$$I(a, a^{-k}) \leqq I(a, a^{-1})+1+[i, -k-i-1]_{-i-1}, \ -i=1, \cdots, \left[\frac{k}{2}\right].$$

Immediately we have (7.1) and (7.2). By (C) there exists a positive integer t_s such that $I(a^s, a^{st_s})-1 \geqq 0$ for any s and

$$I(a^s, a^{st_s})-1=\begin{cases} 0 & \text{if } s=0 \\ [st_s, s-1]_s & \text{if } s \geqq 1 \\ [s, st_s-1]_{-s} & \text{if } s \leqq -1. \end{cases}$$

Thus we have (7.3). The converse of the theorem is obvious.

3. The Case where G is a Direct Product. Suppose that G is the direct product of two commutative groups A and B.

$$G=A \times B=\{(a, b); a \in A, b \in B\}.$$

Let $E(a_1, b_1; a_2, b_2; a_3, b_3)$ denote the equation obtained by setting x, y, and z as (a_1, b_1), (a_2, b_2), and (a_3, b_3) respectively in (B) and let $E'(a_1, b_1; a_2, b_2; a_3, b_3)$ be the one obtained by exchanging the two sides of $E(a_1, b_1; a_2, b_2; a_3, b_3)$ with each other.

Lemma 5. *All of the equations $E(a_1, b_1; a_2, b_2; a_3, b_3)$ are expressed by equations of the types $E(a_1, f; a_2, f; e, b_3)$, $E(e, b_1; e, b_2; a_3, f)$, $E(a_1, b_1; e, b_2; a_3, f)$, $E(a_1, f; a_2, f; a_3, f)$, and $E(e, b_1; e, b_2; e, b_3)$ where e and f are the indentities of A and B respectively.*

Proof. Add $E'(a_1, b_1; e, b_2; a_2, f)$, $E'(a_1a_2, b_1b_2; e, b_3; a_3, f)$, $E(a_2, b_2; e, b_3; a_3, f)$, $E(a_1, b_1; e, b_2b_3; a_2a_3, f)$, $E'(e, b_2; e, b_1; a_1, f)$, $E'(e, b_3; e, b_1b_2; a_1a_2, f)$, $E(e, b_3; e, b_2; a_2, f)$, $E(e, b_2b_3; e, b_1; a_1, f)$, $E'(e, b_3; e, b_2; e, b_1)$, $E'(a_2, f; a_1, f; e, b_1b_2)$, $E'(e, b_2b_3; a_2, f; a_3, f)$, $E(a_2a_3, f; a_1, f; e, b_1b_2b_3)$, $E(e, b_1b_2b_3; a_1a_2, f; a_3, f)$, and $E(a_1, f; a_2, f; a_3, f)$. Then we obtain $E(a_1, b_1; a_2, b_2; a_3, b_3)$.

Lemma 6. *For any $a_1, a_2 \in A$, $b_1, b_2 \in B$ it holds that*

$$I((a_1, b_1), (a_2, b_2))=I((a_1, f), (a_2, f))+I((e, b_1), (e, b_2)) \\ +I((a_1a_2, f), (e, b_1b_2))-I((a_1, f), (e, b_1))-I((a_2, f), (e, b_2)).$$

Proof. Add $E'(a_1, b_1; a_2, f; e, b_2)$, $E'(a_2, f; a_1, f; e, b_1)$, and $E'(e, b_2; e, b_1; a_1a_2, f)$ g.e.d.

Conversely if $I((a_1, f), (e, b_1))$, for all $a_1 \in A$, $b_1 \in B$, are given and if $I((a_1, b_1), (a_2, b_2))$ is defined in this manner, we can easily prove that the function I satisfies (B).

We define I_A and I_B as follows:

$$I_A(a_1, a_2)=I((a_1, f), (a_2, f)), \ I_B(b_1, b_2)=I((e, b_1), (e, b_2))$$

Then we verify that I_A and I_B are I-functions defined on $A'=\{(a, f);\ a\in A\}$ and $B'=\{(e, b);\ b\in B\}$ respectively.

Therefore, by Lemmas 5 and 6, we get the following theorem:

Theorem 5. *Suppose that a direct product $G=A\times B$ of two commutative groups A, B, and that I-values I_A for A and I_B for B are already given. Then the set $I_{A,B}$ of function values $I((a,f), (e,b))$, $a\in A\backslash\{e\}$, $b\in B\backslash\{f\}$, are independent up to relative size considerations and every other value $I((a_1b_1), (a_2, b_2))$ is determined from I_A, I_B, and $I_{A,B}$ by the form in Lemma 6.*

Remark. All elements of $I_{A,B}$ in Theorem 5 must be chosen so as to satisfy $I((a_1, b_1), (a_2, b_2))\geqq 0$ for all (a_1, b_1), (a_2, b_2), and additionally (C). For this, how can we choose $I((a, f), (e, b))$ in advance? The complete solution, namely the theory corresponding to Theorem 2 or Theorem 4, is left to the continued series of this paper.

Let $G=A_1\times\cdots\times A_n$ be the direct product of n commutative groups $A_1, \cdots, A_n$. Suppose that I-values I_i for A_i, $i=1, \cdots, n$ are already given, and consider sets I'_j of function values:

$I'_j((a_1, \cdots, a_{j-1}, e_j, \cdots, e_n), (e_1, \cdots, e_{j-1}, a_j, e_{j+1}, \cdots, e_n))\quad j=2, \cdots, n$

where

$$(a_1, \cdots, a_{j-1}, e_j, \cdots, e_n)\neq(e_1, \cdots, e_n),$$
$$(e_1, \cdots, e_{j-1}, a_j, e_{j+1}, \cdots, e_n)\neq(e_1, \cdots, e_n)$$

and e_k is the identity element of A_k. Then the union of I'_j, $j=2, \cdots, n$, is a set of I-values independent up to relative size considerations and every other value is determined from $I_1, I_2, \cdots, I_n, I'_2, \cdots, I'_n$.

Since every finitely generated commutative group is the direct product of a finite number of cyclic groups, the results obtained above can be applied to any finitely generated commutative group. We have easily the following theorem:

Theorem 6. *If a commutative group G has order n, then the number of "independent" I-function values for G is $n-1$.*

References

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[2] A. H. Clifford and G. B. Preston: The algebraic theory of semigroups I. Surveys 7, Amer. Math. Soc., Providence, R. I. (1961).