122. *Certain Embedding Problems of Semigroups.* II

By Reikichi YOSHIDA
Department of Mathematics, Ritsumeikan University
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In this paper the author will discuss the problems 1, 3 presented by T. Tamura and N. Graham [4]. The terminology and the numbers of formulas in the previous paper will be used here without definitions.

$\Lambda[P]$ denote the left [right] translation semigroup of a semigroup S. The necessary and sufficient condition so that S is embeddable in the right-sided way was given by Theorem 3 in [4]. But we can find subsemigroups A of Theorem 3 in several ways. We wish to rewrite Theorem 3.

Let C be the set of all left translations λ of S such that $\lambda\rho=\rho\lambda$ for all right translations ρ of S, and D the set of all left translations of S which has a linked right translation of S. If we set $\bar{A}=C\cap D$, then we can prove easily that $\bar{A}$ becomes a semigroup containing the identical mapping $\underline{1}$ and the inner left translation semigroup Λ_0.

If λ and ρ are linked, we write $\lambda(\mathrm{LK})\rho$. As in [4], we have

Lemma a. If $\tau\in\Lambda\cap P$, then it follows that $\tau(\mathrm{LK})\tau$.

Moreover by Lemma a,

Lemma b. If S is embeddable in the mixed way, then $D=\Lambda$. Let $P\backslash\bar{A}$ be the set of elements of P which are not in $\bar{A}$.

Theorem 3′. S is embeddable in the right-sided way if and only if there exists a left translation $\bar{\alpha}$ of $\bar{A}$ such that $\bar{\alpha}(\mathrm{LK})\rho$ for all $\rho\in(P\backslash\bar{A})$.

Proof. Let S be an embeddable semigroup in the right-sided way. By Theorem 3, there is a subsemigroup A of Λ such that the conditions (15) and (16) hold. If $\alpha\in A$, then there exists $\rho\in P$ where $\alpha(\mathrm{LK})\rho$, and whence $\alpha\in D$. Also we conclude $\alpha\in C$ from (15). Therefore we see that $A\subseteqq\bar{A}$. For every right translation ρ of S, there exists an element α of $A\subseteqq\bar{A}$ such that $\alpha(\mathrm{LK})\rho$ by using (15).

Conversely, if we take $\bar{A}$ as a subsemigroup A of Theorem 3, then $\bar{A}$ satisfies (15) and (16), since we have $\tau(\mathrm{LK})\tau$ for $\tau\in P\cap\bar{A}$.

Theorem 5. Let S be an embeddable semigroup in the mixed way. Then S is embeddable in the right-sided way if and only if every translation ρ in $P\backslash C$ is linked with some left translation γ in C.

Proof. From Theorem 2, if S is embeddable in the mixed way, then there exist subsemigroups A and B having the properties (11), (12), and (13). Let $\tau\in A\cap B$. Then $\tau\beta=\beta\tau$ for all β in B, and $\alpha\tau=\tau\alpha$ for all $\alpha\in A$, and so τ commute with every left and right translations, since $A\cup B=\Lambda\cup P$. Therefore it follows that $A\cap B\subseteqq C$.

Moreover by Lemma b, $\bar{A}=C\cap D=C\cap\Lambda=C$. Hence we obtain Theorem 5 by replacing C as $\bar{A}$ in Theorem 3′.

Theorem 6. Assume that S is embeddable in the right-sided way. If $\Lambda\backslash P\subseteqq\bar{A}$, then S is embeddable in the mixed way.

Proof. We set $A=\bar{A}$ and $B=P$. Then $A\cup B=\bar{A}\cup P=\Lambda\cup P$, that is the condition (13) holds. By Theorem 3′, every right translation $\rho\in B=P$ is linked with some $\bar{\alpha}$ in $\bar{A}$. Furthermore, since $\bar{\alpha}\in D$, there exists a right translation ρ linked with $\bar{\alpha}$. The condition (12) follows from $\bar{A}\subseteqq C$.

Theorem 7. Suppose that S is embeddable semigroup in the right-sided way. Then S is embeddable in the two-sided way if and only if $\bar{A}=\Lambda$.

Proof. If $\bar{A}=\Lambda$, then $\Lambda=C$, and $\Lambda=D$. Since S is embeddable in the right-sided way, every right translation ρ is linked with some $\bar{\alpha}$ in $\bar{A}=\Lambda$ by Theorem 3′. Whence S is embeddable in the two-sided way.

Conversely, if S is embeddable in the two-sided way, then it follows that $C=\Lambda=D$, and evidently $\bar{A}=C\cap D=\Lambda$.

Finally, we are concerned with Problem 3 in [4]—can any semigroup be embeddable either in the right-sided way or in the left-sided way?

	e	f	g	a	0
e	e	a	e	a	0
f	0	f	g	0	0
g	g	f	g	f	0
a	0	a	e	0	0
0	0	0	0	0	0

A counter example is provided by the semigroup $S=\{e, f, g, a, 0\}$ in p. 8 of [1]. $\lambda_1=\begin{pmatrix} efga0 \\ e00a0 \end{pmatrix}$, $\lambda_2=\begin{pmatrix} efga0 \\ g00f0 \end{pmatrix}$, and $\lambda_3=\begin{pmatrix} efga0 \\ gaef0 \end{pmatrix}$ are left translations of S (cf. [2]). But they have not any linked right translation. Hence S does not satisfy the necessary condition in order that S is embeddable in the left-sided way. Also $\rho_1=\begin{pmatrix} efga0 \\ 0f0a0 \end{pmatrix}$, $\rho_2=\begin{pmatrix} efga0 \\ agfe0 \end{pmatrix}$, and $\rho_3=\begin{pmatrix} efga0 \\ 0g0e0 \end{pmatrix}$ are right translations of S having no linked left translation, respectively. Therefore S is not embeddable in the right-sided way, too.

References

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