121. *On the Gibbs Phenomenon for* (K, 1) *Means*

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§ 1. Zygmund [9] introduced the following method of summability which is similar to the *Lebesgue method* $(R, 1)$[1]: When a series

$$(1) \qquad \sum_{n=0}^{\infty} u_n$$

is given, if

(i) the series

$$\frac{2}{\pi}\sum_{n=1}^{\infty} u_n \int_h^{\pi} \frac{\sin nt}{2\tan\frac{1}{2}t}\,dt$$

converges for small positive h, and if

(ii) the limit of

$$u_0+\frac{2}{\pi}\sum_{n=1}^{\infty} u_n \int_h^{\pi} \frac{\sin nt}{2\tan\frac{1}{2}t}\,dt$$

for $h\to+0$ exists and equals s. then he calls that the series (1) is *summable* $(K, 1)$ to s.

The convergence of (1) need not imply its summability $(K, 1)$ as well as in the case of the method $(R, 1)$. We shall study, in this note, the Gibbs phenomenon of the Fourier series

$$(2) \qquad \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

for the $(K, 1)$ means. Ching-Hsi Lee [5] proved the following

Theorem 1. *The* $(R, 1)$ *means of the series* (2) *does not present the Gibbs phenomenon at* $x=0$.

We shall prove here the following

Theorem 2. *The* $(K, 1)$ *means of the series* (2) *does not present the Gibbs phenomenon at* $x=0$.

Proof. Let

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1) We say that the series (1) is summable $(R, 1)$ to s, if $\sum_{n=1}^{\infty} u_n \frac{\sin nh}{nh}$ converges for small positive h, and if $\lim_{h\to+0}\left\{u_0+\sum_{n=1}^{\infty} u_n \frac{\sin nh}{nh}\right\}=s$. See, e.g., Hardy [1], p. 89, Zeller [8], p. 158.

$$T(h, x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n}\int_h^\pi \frac{\sin nt}{2\tan\frac{1}{2}t}\,dt.$$

Then we have

$$\begin{aligned} T(h, x) &= \frac{2}{\pi}\sum_{n=1}^{\infty}\int_h^\pi \frac{\sin nx \sin nt}{2n\tan\frac{1}{2}t}\,dt \\ &= \frac{1}{\pi}\int_h^\pi \sum_{n=1}^{\infty}\frac{\sin nx \sin nt}{n}\cot\frac{1}{2}t\,dt \\ &= \frac{1}{2\pi}\int_h^\pi \cot\frac{1}{2}t \log\frac{\sin\frac{1}{2}(x+t)}{\sin\frac{1}{2}|x-t|}\,dt, \end{aligned}$$

where the termwise integration is justified as in a similar case in Hardy and Rogosinski [2], p. 178. Since

$$\log\frac{\sin\frac{1}{2}(x+t)}{\sin\frac{1}{2}|x-t|}>0$$

for $0<x<\pi$ and $0<t<\pi$, we obtain

$$\begin{aligned} 0 \le T(h, x) &\le \frac{1}{2\pi}\int_0^\pi \cot\frac{1}{2}t\log\frac{\sin\frac{1}{2}(x+t)}{\sin\frac{1}{2}|x-t|}\,dt \\ &= \frac{1}{2}(\pi-x)<\frac{\pi}{2} \end{aligned}$$

again by termwise integration. This proves our assertion.

§ 2. It is convenient to assume

$$s_0=u_0=0\ ,$$
$$s_n=u_1+u_2+\cdots+u_n,\ n=1, 2, \cdots.$$

We then obtain

$$\begin{aligned} &\frac{2}{\pi}\sum_{n=1}^{N}u_n\int_h^\pi \frac{\sin nt}{2\tan\frac{1}{2}t}\,dt \\ (3)\qquad &= \frac{2}{\pi}\sum_{n=1}^{N-1}s_n\int_n^\pi \frac{\sin nt-\sin(n+1)t}{2\tan\frac{1}{2}t}\,dt \\ &\quad + s_N\frac{2}{\pi}\int_h^\pi \frac{\sin Nt}{2\tan\frac{1}{2}t}\,dt \end{aligned}$$

by partial summation. If we assume

(4) $$s_n=o(n) \quad \text{as} \quad n\to\infty\ ,$$

then we get, from (3),

$$
\begin{aligned}
&\frac{2}{\pi}\sum_{n=1}^{\infty}u_n\int_h^{\pi}\frac{\sin nt}{2\tan\frac{1}{2}t}\,dt \\
&=\frac{2}{\pi}\sum_{n=1}^{\infty}s_n\int_h^{\pi}\frac{\sin nt-\sin(n+1)t}{2\tan\frac{1}{2}t}\,dt \\
&=\frac{1}{\pi}\sum_{n=1}^{\infty}s_n\left\{\frac{\sin nh}{n}+\frac{\sin(n+1)h}{n+1}\right\} \\
&=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{s_n+s_{n-1}}{2}\,\frac{\sin nh}{n}.
\end{aligned}
\tag{5}
$$

When a sequence $\{s_n\}$ is given, the sequence-to-sequence transformation (Y) is defined by means of the equation

$$y_n=\frac{1}{2}(s_n+s_{n-1}), \qquad n=0,1,\cdots,$$

where $s_{-1}=0$. As is easily seen, this transformation is regular. For this transformation, see, e.g., Ishiguro [3], Szász [7].

From (5) we get immediately the following

Theorem 3. *If*

$$s_n=o(n),$$

the methods $(K,1)$ and $(R_1)\cdot Y$[2] *are equivalent,*[3] *where $(R_1)\cdot Y$ is the iteration product of these two methods.*

When (1) is a Fourier series, we see easily $s_n=o(n)$. It is interesting to note that if $\{s_n\}$ is summable (Y) then $s_n=o(n)$. See, e.g., Szász [7], p. 8.

§ 3. We now study, by the last theorem, the Lebesgue constants for the transformation $(K,1)$ following the lines of Szász [6]. We assume that $f(t)$ is integrable and that $|f(t)|\leq 1$, $0\leq t\leq\pi$. Let

$$s_n=s_n(x)=\sum_{\nu=1}^{n}a_\nu\cos\nu x=\frac{1}{2}a_n\cos nx+y_n(x),^{4)}$$

where $y_n(x)$ is the n-th means (Y) of $\{s_n(x)\}$. Then

$$y_n(x)=\frac{1}{\pi}\int_0^{\pi}\phi(t)\cot\frac{1}{2}t\sin nt\,dt,$$

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2) We say that the series (1) is *summable* (R_1) to s, if $\sum_{n=1}^{\infty}s_n\frac{\sin nh}{n}$ converges for small positive h, and if $\lim_{h\to+0}\frac{2}{\pi}\sum_{n=1}^{\infty}s_n\frac{\sin nh}{n}=s$. See, e.g., Hardy and Rogosinski [2], Kuttner [4], Szász [6], Zeller [8], p. 158.

3) Given two summability methods A, B, we say that A *implies* B if any series or sequence summable A is summable B to the same sum. We say that A and B are *equivalent* if A implies B and B implies A.

4) We usually use the notation '$s_n^*(x)$' in place of '$y_n(x)$'.

where

$$\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}.$$

Because of periodicity, we may restrict ourselves, as usual, to $x=0$. Now we have, from Hardy and Rogosinski [2],

$$\sum_{n=1}^{\infty} y_n(0)\frac{\sin nh}{n}=\frac{1}{\pi}\int_0^{\pi} f(t)\cot\frac{1}{2}t\sum_{n=1}^{\infty}\frac{\sin nt \sin nh}{n}\,dt,$$

where, if $0<t<\pi$ and $0<h<\pi$,

$$\sum_{n=1}^{\infty}\frac{\sin nt\sin nh}{n}=\frac{1}{2}\log\frac{\sin\frac{1}{2}(t+h)}{\sin\frac{1}{2}|t-h|}>0.$$

It now follows that

$$\begin{aligned}\left|\sum_{n=1}^{\infty} y_n(0)\frac{\sin nh}{n}\right| &\leq \frac{1}{2\pi}\int_0^{\pi}\cot\frac{1}{2}t\log\frac{\sin\frac{1}{2}(t+h)}{\sin\frac{1}{2}|t-h|}\,dt\\ &=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin nh}{n}\int_0^{\pi}\cot\frac{1}{2}t\sin nt\,dt\\ &=\sum_{n=1}^{\infty}\frac{\sin nh}{n}=\frac{\pi-h}{2}\leq\frac{\pi}{2},\end{aligned}$$

where the termwise integration is legitimate as in Hardy and Rogosinski [2], p. 178.

This proves the following

Theorem 4. *The Lebesgue constants for the method $(K, 1)$ are uniformly bounded.*

References

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