# 118. Some Applications of the Functional Representations of Normal Operators in Hilbert Spaces. XVI

By Sakuji INOUE
Faculty of Education, Kumamoto University
(Comm. by Kinjirô KUNUGI, M.J.A., Sept. 13, 1965)

Let $\{\lambda_\nu\}_{\nu=1,2,3,\ldots}$, $D_j (j=1, 2, 3, \cdots, n)$, and $T(\lambda)$ be the same notations as those defined in Part XIII [cf. Proc. Japan Acad., Vol. 40, No. 7, 492–497 (1964)]; let $\chi(\lambda)$ be the sum of the first and second principal parts of $T(\lambda)$; and let us suppose that $\{\lambda_\nu\}$ is everywhere dense on a (closed or open) rectifiable Jordan curve $\Gamma$ and that for any small positive $\varepsilon$ the circle $|\lambda|=\sup_\nu|\lambda_\nu|+\varepsilon$ contains the mutually disjoint sets $\Gamma, D_1, D_2, \cdots, D_{n-1}$, and $D_n$ inside itself. In this paper we shall discuss the respective behaviours concerning $\rho$ of the maximum moduli of $\chi(\lambda)$ and $T(\lambda)$ on the circle $|\lambda|=\rho$ with $\sup_\nu|\lambda_\nu|<\rho<\infty$.

Theorem 43. Let $T(\lambda)$ be the function with singularities $\overline{\{\lambda_\nu\}}\cup\left[\bigcup_{j=1}^{n} D_j\right]$ stated above; let $\chi(\lambda)$ be the sum of the first and second principal parts of $T(\lambda)$; let $\sigma=\sup_\nu|\lambda_\nu|$; and let $M_\chi(\rho)$ denote the maximum modulus of $\chi(\lambda)$ on the circle $|\lambda|=\rho$ with $\sigma<\rho<\infty$. Then

$$M_\chi(\rho')\leqq M_\chi(\rho) \qquad (\sigma<\rho<\rho'<\infty),$$
$$M_\chi(\rho)\to\infty \qquad (\rho\to\sigma),$$

and for any $\rho$ with $\sigma<\rho<\infty$

$$\text{(A)}\quad \frac{1}{2}\sqrt{\sum_{\mu=1}^{\infty}|a_\mu(\rho)+ib_\mu(\rho)|^2}\leqq M_\chi(\rho)\leqq\frac{1}{2}\sum_{\mu=1}^{\infty}|a_\mu(\rho)+ib_\mu(\rho)|<\infty,$$

where

$$\left.\begin{aligned} a_\mu(\rho)&=\frac{1}{\pi}\int_0^{2\pi}T(\rho e^{it})\cos\mu t\,dt\\ b_\mu(\rho)&=\frac{1}{\pi}\int_0^{2\pi}T(\rho e^{it})\sin\mu t\,dt\end{aligned}\right\}(\sigma<\rho<\infty,\ \mu=1, 2, 3, \cdots).$$

Proof. Let $C$ denote the positively oriented circle $|\lambda|=\rho$ with $\sigma<\rho<\infty$, and let $R(\lambda)$ be the ordinary part of $T(\lambda)$. Then, as already demonstrated in Theorem 30 of Part XIII quoted above,

$$\frac{1}{2\pi i}\int_C\frac{T(\lambda)}{(\lambda-z)^k}d\lambda=\begin{cases}R^{(k-1)}(z)/(k-1)! & \text{(for every } z \text{ inside } C)\\ -\chi^{(k-1)}(z)/(k-1)! & \text{(for every } z \text{ outside } C),\end{cases}$$

where $k=1, 2, 3, \cdots$. Furthermore, as can be seen from the method of the proof of (5) [cf. Proc. Japan Acad., Vol. 38, No. 8, 452–456 (1962)], it is verified with the help of these relations that

$$R(\kappa\rho e^{i\theta})+\chi\Big(\frac{\rho}{\kappa}e^{i\theta}\Big)=\frac{1}{2\pi}\int_0^{2\pi}T(\rho e^{it})\frac{1-\kappa^2}{1+\kappa^2-2\kappa\cos(\theta-t)}dt$$
$$(\sigma<\rho<\infty,\ 0<\kappa<1).$$

Since, on the other hand,

$$R(\kappa\rho e^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}R(\rho e^{it})\frac{1-\kappa^2}{1+\kappa^2-2\kappa\cos(\theta-t)}dt$$

by the definition that $R(\lambda)$ is an integral function, we have therefore

$$\chi\Big(\frac{\rho}{\kappa}e^{i\theta}\Big)=\frac{1}{2\pi}\int_0^{2\pi}\chi(\rho e^{it})\frac{1-\kappa^2}{1+\kappa^2-2\kappa\cos(\theta-t)}dt.$$

This result and the equality

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1-\kappa^2}{1+\kappa^2-2\kappa\cos(\theta-t)}dt=1$$

enable us to establish the desired inequality $M_\chi\Big(\frac{\rho}{\kappa}\Big)\leqq M_\chi(\rho)$ for every $\kappa$ with $0<\kappa<1$.

In addition to it, the following equalities hold:

$$(36)\qquad T\Big(\frac{\rho}{\kappa}e^{i\theta}\Big)=\frac{a_0(\rho)}{2}+\frac{1}{2}\sum_{\mu=1}^{\infty}(a_\mu(\rho)-ib_\mu(\rho))\Big(\frac{e^{i\theta}}{\kappa}\Big)^\mu$$
$$+\frac{1}{2}\sum_{\mu=1}^{\infty}(a_\mu(\rho)+ib_\mu(\rho))\Big(\frac{\kappa}{e^{i\theta}}\Big)^\mu\qquad(0<\kappa<1),$$

$$\frac{1}{2}\sum_{\mu=1}^{\infty}(a_\mu(\rho)+ib_\mu(\rho))\Big(\frac{\kappa}{e^{i\theta}}\Big)^\mu=\chi\Big(\frac{\rho}{\kappa}e^{i\theta}\Big)\qquad(0<\kappa<1),$$

and

$$\frac{a_0(\rho)}{2}+\frac{1}{2}\sum_{\mu=1}^{\infty}(a_\mu(\rho)-ib_\mu(\rho))\Big(\frac{e^{i\theta}}{\kappa}\Big)^\mu=R\Big(\frac{\rho}{\kappa}e^{i\theta}\Big)\qquad(0<\kappa<\infty),$$

where $a_\mu(\rho)$ and $b_\mu(\rho)$ are the coefficients defined in the statement of the present theorem and $a_0(\rho)=\frac{1}{\pi}\int_0^{2\pi}T(\rho e^{it})dt$ [cf. Proc. Japan Acad., Vol. 40, No. 8, 654–659 (1964)]. Accordingly

$$\Big|\chi\Big(\frac{\rho}{\kappa}e^{i\theta}\Big)\Big|\leqq\frac{1}{2}\sum_{\mu=1}^{\infty}|a_\mu(\rho)+ib_\mu(\rho)|\kappa^\mu$$

and

$$(37)\qquad \frac{1}{2\pi}\int_0^{2\pi}\Big|\chi\Big(\frac{\rho}{\kappa}e^{it}\Big)\Big|^2dt=\frac{1}{4}\sum_{\mu=1}^{\infty}|a_\mu(\rho)+ib_\mu(\rho)|^2\kappa^{2\mu},$$

so that

$$(38)\qquad \frac{1}{2}\sqrt{\sum_{\mu=1}^{\infty}|a_\mu(\rho)+ib_\mu(\rho)|^2\kappa^{2\mu}}\leqq M_\chi\Big(\frac{\rho}{\kappa}\Big)\leqq\frac{1}{2}\sum_{\mu=1}^{\infty}|a_\mu(\rho)+ib_\mu(\rho)|\kappa^\mu$$
$$(0<\kappa<1).$$

Next we can find immediately from (36) that

$$a_\mu\Big(\frac{\rho}{\kappa}\Big)+ib_\mu\Big(\frac{\rho}{\kappa}\Big)=\frac{1}{\pi}\int_0^{2\pi}T\Big(\frac{\rho}{\kappa}e^{it}\Big)e^{i\mu t}dt$$
$$=((a_\mu(\rho)+ib_\mu(\rho))\kappa^\mu$$

and hence that

$$\sum_{\mu=1}^{\infty}\left|a_\mu\left(\frac{\rho}{\kappa}\right)+ib_\mu\left(\frac{\rho}{\kappa}\right)\right|=\sum_{\mu=1}^{\infty}|a_\mu(\rho)+ib_\mu(\rho)|\,\kappa^\mu.$$

On the other hand, as already shown in Theorem 38 [cf. Proc. Japan Acad., Vol. 40, No. 8, 654–659 (1964)],

$$\chi(\lambda)=\sum_{\mu=1}^{\infty}C_{-\mu}\lambda^{-\mu}\quad(|\lambda|\geqq\rho>\sigma),$$

where

$$C_{-\mu}=\frac{1}{2\pi i}\int_\sigma\frac{T(\lambda)}{\lambda^{-\mu+1}}d\lambda$$
$$=\frac{\rho^\mu(a_\mu(\rho)+ib_\mu(\rho))}{2}$$

and $\sum_{\mu=1}^{\infty}C_{-\mu}\lambda^{-\mu}$ is essentially an infinite series. Since, in addition, it is verified by Cauchy's integral theorem that $C_{-\mu}$ is irrespective of $\rho$, $\sum_{\mu=1}^{\infty}C_{-\mu}\lambda^{-\mu}$ is absolutely convergent in the domain $\{\lambda\colon|\lambda|>\sigma\}$. Hence

$$\sum_{\mu=1}^{\infty}\left|a_\mu\left(\frac{\rho}{\kappa}\right)+ib_\mu\left(\frac{\rho}{\kappa}\right)\right|<\sum_{\mu=1}^{\infty}|a_\mu(\rho)+ib_\mu(\rho)|<\infty$$

for any $\rho$ with $\sigma<\rho<\infty$. By allowing $\kappa$ in (38) to tend to 1 we obtain therefore (A), as we were to prove.

Thus it remains only to prove that $M_\chi(\rho)\to\infty$ $(\rho\to\sigma)$.

Since there exists on the circle $|\lambda|=\sigma$ at least one point of $\{\lambda_\nu\}$ or one accumulating point of $\{\lambda_\nu\}$ such that it does not belong to $\{\lambda_\nu\}$ itself, we denote by $\sigma e^{i\alpha}$ that singularity of $\chi(\lambda)$. If, contrary to what we wish to prove, $\{M_\chi(\rho)\}_\rho$ were bounded on an open interval $(\sigma,\sigma')$ with $\sigma<\sigma'<\infty$, then $\chi(\lambda)$ would be bounded on the intersection of the annular domain $\{\lambda\colon\sigma<|\lambda|<\sigma'\}$ and an arbitrary neighbourhood of $\sigma e^{i\alpha}$. Since, however, $\chi(\lambda)$ is regular and hence continuous on that intersection, the above result is in contradiction with the hypothesis that every $\lambda_\nu$ of $\{\lambda_\nu\}$ everywhere dense on $\Gamma$ is a pole in the sense of the functional analysis. Consequently $M_\chi(\rho)\to\infty$ $(\rho\to\sigma)$, as we wished to prove.

Remark 1. The notation $\overline{\{\lambda_\nu\}}$ in the statement of Theorem 43 denotes the closure of $\{\lambda_\nu\}$.

Theorem 44. Let $T(\lambda)$ and $\sigma$ be the same notations as those in Theorem 43, and $M_T(\rho)$ the maximum modulus of $T(\lambda)$ on the circle $|\lambda|=\rho$ with $\sigma<\rho<\infty$. If the ordinary part $R(\lambda)$ of $T(\lambda)$ is a constant, then

$$M_T(\rho')\leqq M_T(\rho)\qquad(\sigma<\rho<\rho'<\infty),$$
$$M_T(\rho)\to\infty\qquad(\rho\to\sigma);$$

and if, contrary to it, $R(\lambda)$ is an integral function, then there exists a suitable positive constant $\rho_0$ such that the inequality $M_T(\rho')\leqq M_T(\rho)$

holds for every pair of positive constants $\rho$, $\rho'$ with $\sigma<\rho<\rho'<\rho_0<\infty$, and also $M_T(\rho)\to\infty$ $(\rho\to\sigma)$.

Proof. If $R(\lambda)$ is a constant $c$, then, as can be found immediately from the earlier discussion,

$$T\left(\frac{\rho}{\kappa}e^{i\theta}\right)=\frac{1}{2\pi}\int_0^{2\pi}T(\rho e^{it})\frac{1-\kappa^2}{1+\kappa^2-2\kappa\cos(\theta-t)}dt \quad (\sigma<\rho<\infty,\ 0<\kappa<1)$$

and hence $M_T(\rho')\leqq M_T(\rho)$ for every pair of $\rho$, $\rho'$ with $\sigma<\rho<\rho'<\infty$. Since, moreover, $T(\lambda)=c+\chi(\lambda)$, $M_T(\rho)\geqq M_\chi(\rho)-|c|$. Thus it is a direct consequence of the preceding theorem that $M_T(\rho)\to\infty$ $(\rho\to\sigma)$.

We next consider the case where $R(\lambda)$ is a polynomial in $\lambda$ or a transcendental integral function.

In the first place it follows from the equality $T(\lambda)=R(\lambda)+\chi(\lambda)$ that

$$M_\chi(\rho)-M_R(\rho)\leqq M_T(\rho)\leqq M_\chi(\rho)+M_R(\rho) \quad (\sigma<\rho<\infty,\ M_R(\rho)=\max_\theta|R(\rho e^{i\theta})|). \tag{39}$$

On the other hand, since $M_\chi(\rho)\to\infty$ $(\rho\to\sigma)$ and since $M_R(\rho)$ decreases monotonously as $\rho$ tends to $\sigma$, there exist suitable positive constants $\rho_0$, $\rho_0'$ with $\sigma<\rho_0<\rho_0'<\infty$ such that $M_\chi(\rho_0')+M_R(\rho_0')\leqq M_\chi(\rho_0)-M_R(\rho_0)$ and hence $M_T(\rho_0')\leqq M_T(\rho_0)$. In fact, the inequality $M_\chi(\rho_0')+M_R(\rho_0')\leqq M_\chi(\rho_0)-M_R(\rho_0)$ holds provided that $\rho_0$ is chosen suitably near to $\sigma$ in comparison with $\rho_0'$. Suppose now that the annular closed domains $\{\lambda: \rho_0\leqq|\lambda|\leqq\rho_0'\}$ and $\{\lambda: \rho\leqq|\lambda|\leqq\rho_0'\}$ with $\sigma<\rho<\rho_0<\rho_0'<\infty$ are mapped by the transformation $w=T(\lambda)$ onto $\Delta(\rho_0,\rho_0')$ and $\Delta(\rho,\rho_0')$ in the complex $w$-plane respectively. Then $\Delta(\rho_0,\rho_0')\subset\Delta(\rho,\rho_0')$ and $\Delta(\rho,\rho_0')$ here enlarges outwards as $\rho$ decreases to $\sigma$. As a result, it is found from the principle of the maximum modulus for a regular function that $M_T(\rho_0)\leqq M_T(\rho)$ for every $\rho$ with $\sigma<\rho<\rho_0$. By following the argument used above, we can therefore establish the inequality $M_T(\rho')\leqq M_T(\rho)$ holding for every pair of $\rho$, $\rho'$ with $\sigma<\rho<\rho'<\rho_0$. Furthermore it is at once obvious from (39) and the preceding theorem that $M_T(\rho)\to\infty$ $(\rho\to\sigma)$.

With these results the present theorem has been proved.

Remark 2. The results in Theorems 43 and 44 are valid, of course, for the function $\hat{T}(\lambda)$ treated in Theorems 41 and 42 [cf. Proc. Japan Acad., Vol. 41, No. 2, 150–154 (1965)].

On the assumption that $R(\lambda)$ is not a constant, we shall next treat of the relation among $M_R(\rho)$, $M_\chi(\rho)$, and the coefficients of $a_\mu$, $b_\mu$ $(\mu=1,2,3\cdots)$. The following theorem concerning it is, in particular, significant for sufficiently large values of $\rho$ from a point of view of the fact that

$$\left.\begin{matrix}M_R(\rho)\to\infty\\ M_\chi(\rho)\to 0\end{matrix}\right\}\quad(\rho\to\infty).$$

Theorem 45. Let $T(\lambda)$ and $\sigma$ be the same notations as before; and let the ordinary part $R(\lambda)$ of $T(\lambda)$ be an integral function, not a constant; and let $K_\mu=a_\mu^2(\rho)+b_\mu^2(\rho)$, where $a_\mu(\rho)$ and $b_\mu(\rho)$ are the coefficients defined before. Then $K_\mu$ is independent of values of $\rho$ as far as $\rho$ satisfies the condition $\sigma<\rho<\infty$; and moreover

$$\left|\sum_{\mu\geqq 1} K_\mu\right| \leqq 4M_R(\rho)M_\chi(\rho) \qquad (\text{for every } \rho \text{ with } \sigma<\rho<\infty),$$

where $\sum_{\mu\geqq 1} K_\mu$ is a finite or an infinite series according as $R(\lambda)$ is a polynomial in $\lambda$ or a transcendental integral function.

Proof. As already pointed out, $a_\mu\left(\frac{\rho}{\kappa}\right)+ib_\mu\left(\frac{\rho}{\kappa}\right)=(a_\mu(\rho)+ib_\mu(\rho))\kappa^\mu$ $(\sigma<\rho<\infty,\ 0<\kappa<1)$; and similarly $a_\mu\left(\frac{\rho}{\kappa}\right)-ib_\mu\left(\frac{\rho}{\kappa}\right)=(a_\mu(\rho)-ib_\mu(\rho))\kappa^{-\mu}$. Accordingly $a_\mu^2\left(\frac{\rho}{\kappa}\right)+b_\mu^2\left(\frac{\rho}{\kappa}\right)=a_\mu^2(\rho)+b_\mu^2(\rho)$ for every $\rho$ with $\sigma<\rho<\infty$ and for every $\kappa$ with $0<\kappa<1$. This result implies that $a_\mu^2(\rho)+b_\mu^2(\rho)$ is a constant independent of values of $\rho$ as far as $\rho$ satisfies the condition $\sigma<\rho<\infty$. If we now denote by $K_\mu$ this constant depending only on $\mu$, an application of the expansions of $R\left(\frac{\rho}{\kappa}e^{i\theta}\right)$ and $\chi\left(\frac{\rho}{\kappa}e^{i\theta}\right)$ yields the equality

$$\frac{1}{4}\sum_{\mu\geqq 1} K_\mu=\frac{1}{2\pi}\int_0^{2\pi} R\left(\frac{\rho}{\kappa}e^{it}\right)\chi\left(\frac{\rho}{\kappa}e^{it}\right)dt \qquad (\sigma<\rho<\infty,\ 0<\kappa<1)$$

and hence the inequality

$$\frac{1}{4}\left|\sum_{\mu\geqq 1} K_\mu\right| \leqq M_R(\rho)M_\chi(\rho) \qquad (\sigma<\rho<\infty),$$

on the assumption that $R(\lambda)$ is not a constant. Since, in addition,

$$\begin{aligned}\frac{a_\mu(\rho)-ib_\mu(\rho)}{2}&=\frac{1}{2\pi}\int_0^{2\pi} T(\rho e^{it})e^{-i\mu t}dt\\&=\frac{\rho^\mu}{2\pi i}\int_C \frac{T(\lambda)}{\lambda^{\mu+1}}d\lambda\\&=\frac{\rho^\mu R^{(\mu)}(0)}{\mu!},\end{aligned}$$

it is found that $\sum_{\mu\geqq 1} K_\mu$ is a finite series as far as $R(\lambda)$ is a polynomial in $\lambda$. Since, however, the expansion of $R\left(\frac{\rho}{\kappa}e^{i\theta}\right)$ is an infinite series provided that $R(\lambda)$ is a transcendental integral function, and since the expansion of $\chi\left(\frac{\rho}{\kappa}e^{i\theta}\right)$ is also an infinite series as already indicated, $\sum_{\mu\geqq 1} K_\mu$ is an infinite series under this condition on $R(\lambda)$.

The proof of the theorem has thus been finished.

Remark 3. Even if $R(\lambda)$ is a transcendental integral function, $\sum_{\mu\geqq 1}|K_\mu|$ also converges by virtue of the fact that the expansions of $R(\lambda)$ and $\chi(\lambda)$ both converge absolutely in the domain $\{\lambda: \sigma<|\lambda|<\infty\}$. In addition, the result of Theorem 45 is rewritten in the form of

$$\left|\sum_{\mu\geqq 1}\frac{R^{(\mu)}(0)C_{-\mu}}{\mu!}\right|\leqq M_R(\rho)M_\chi(\rho) \qquad (\sigma<\rho<\infty),$$

where $C_{-\mu}$ is the notation used in the course of the proof of Theorem 43.