113. On a Theorem of G. Pólya

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Let a_n $(n=0, 1, 2, \cdots)$ be a sequence of algebraic integers. In 1920 G. Pólya [2] proved that if $\sum_{n=0}^{\infty} na_n z^n$ is a rational function of z, then so is $\sum_{n=0}^{\infty} a_n z^n$. This result has recently been generalized by D. G. Cantor [1], who showed that if $f(x)$ is a non-zero polynomial in x with arbitrary complex coefficients and if $\sum_{n=0}^{\infty} f(n)a_n z^n$ is a rational function, then $\sum_{n=0}^{\infty} a_n z^n$ is again a rational function. In the present note we shall prove the following theorem which is a generalization of the above result due to Pólya in another direction:

Theorem. *Let a_n $(n=0, 1, 2, \cdots)$ be a sequence of numbers belonging to a fixed module over the ring of rational integers with a finite basis in the field of complex numbers. If $\sum_{n=0}^{\infty} na_n z^n$ is a rational function, then so is also $\sum_{n=0}^{\infty} a_n z^n$.*

It is quite easy to see that if the a_n are algebraic integers and if $\sum_{n=0}^{\infty} na_n z^n$ is a rational function, then there exists a finite algebraic extension k of the field of rational numbers such that the ring $\mathfrak{o}(k)$ of algebraic integers of k contains all of the a_n; and, as is well known, the ring $\mathfrak{o}(k)$ has as a module a finite basis over the ring of rational integers.

1. Lemmas. Let K_1 be an arbitrary field of characteristic 0 and K_2 a field containing K_1. We require the following two lemmas which are substantially proved in [2; pp. 4–5].

Lemma 1. *Let $A(z)$ be a non-zero polynomial of $K_1[z]$ and write*

$$A(z)=(P_1(z))^{e_1} \cdots (P_r(z))^{e_r},$$

where $P_1(z), \cdots, P_r(z)$ are distinct irreducible polynomials in $K_1[z]$ and $e_1, \cdots, e_r$ are positive integers. If $B(z)$ is a polynomial of $K_2[z]$, then we have

$$\frac{B(z)}{A(z)}=\sum_{j=1}^{r} \frac{B_j(z)}{(P_j(z))^{e_j}}$$

for some polynomials $B_1(z), \cdots, B_r(z)$ of $K_2[z]$.

Proof. Clear.

Lemma 2. *Let $P(z)$ be an irreducible polynomial of $K_1[z]$ and $Q(z)$ be a polynomial of $K_2[z]$. Let e be a positive integer. Then there exist a rational function $\phi(z)$ of $K_2(z)$ and a polynomial $R(z)$ of $K_2[z]$ with* $\deg R(z)<\deg P(z)$ *such that*

$$\frac{Q(z)}{(P(z))^e}=\frac{d}{dz}\phi(z)+\frac{R(z)}{P(z)}.$$

Proof. The result is obvious for $e=1$. Suppose that the lemma is true for $e=e$. Since K_1, and hence K_2, is assumed to be of characteristic 0, $P(z)$ and $P'(z)$ are relatively prime as polynomials of $K_2[z]$ and we can find two polynomials $S(z)$ and $T(z)$ in $K_2[z]$ satisfying

$$S(z)P(z)+T(z)P'(z)=Q(z).$$

Define the polynomials $H(z)$ and $Q_1(z)$ of $K_2[z]$ by the relations:

$$T(z)=-eH(z),\quad S(z)=H'(z)+Q_1(z).$$

Then we have

$$Q(z) = (H'(z) + Q_1(z))P(z) - eH(z)P'(z) ,$$

whence

$$\frac{Q(z)}{(P(z))^{e+1}}=\frac{d}{dz}\left(\frac{H(z)}{(P(z))^e}\right)+\frac{Q_1(z)}{(P(z))^e}.$$

Thus the lemma is true for $e=e+1$. Our proof is now complete by induction.

2. Proof of the theorem. We denote by R the field of rational numbers, by Z the ring of rational integers, and by M a Z-module with a finite basis $(\xi_1, \cdots, \xi_m)$ in the field of complex numbers. Suppose $a_n \in M$ $(n=0, 1, 2, \cdots)$. Then a_n can be written uniquely in the form

$$a_n=u_{1,n}\xi_1+\cdots+u_{m,n}\xi_m$$

with $u_{1,n}, \cdots, u_{m,n}$ in Z.

Let K be the field obtained from R by adjoining the complex numbers $\xi_1, \cdots, \xi_m$. We distinguish two cases according as K is or is not algebraic over R.

Case 1: K is algebraic over R. In this case K is a finite algebraic extension of R and $\xi_1, \cdots, \xi_m$ are algebraic numbers. There exists, therefore, a non-zero rational integer a such that the numbers $a\xi_1, \cdots, a\xi_m$ are all algebraic integers in K, so that aa_n $(n=0, 1, 2, \cdots)$ are algebraic integers. The theorem follows from the original result of Pólya if we simply replace there a_n by aa_n for each n.

Case 2: K is not algebraic over R. Then K is of the form

$$K=R(\sigma_1, \cdots, \sigma_s, \tau),$$

where $\sigma_1, \cdots, \sigma_s$ $(s\geqq 1)$ are complex numbers which are algebraically independent over R and τ is a complex number which is algebraic over the purely transcendental extension

$$K_0=R(\sigma_1, \cdots, \sigma_s)$$

of R. We may assume without loss of generality that τ is integral over the polynomial ring $R[\sigma_1, \cdots, \sigma_s]$.

In what follows we shall use the abbreviation σ for the set $\sigma_1, \cdots, \sigma_s$: thus, for example, $K=R(\sigma, \tau)$.

The generators $\xi_1, \cdots, \xi_m$ of the module M can now be written as rational functions of σ and τ. In fact, we have

$$\xi_k = \frac{X_k(\sigma, \tau)}{X(\sigma)} \qquad (k=1, \cdots, m),$$

where $X_k(\sigma, \tau) \in R[\sigma, \tau]$ $(k=1, \cdots, m)$ and $X(\sigma) \in R[\sigma]$.

Suppose now that the function $\sum_{n=0}^{\infty} na_n z^n$ be rational. This is equivalent to suppose that the function $\sum_{n=1}^{\infty} na_n z^{n-1}$ be rational, and so there are a non-zero polynomial $A(z)$ in $K_0[z]$ and a polynomial $B(z)$ in $K[z]$ such that

$$\sum_{n=1}^{\infty} na_n z^{n-1} = \frac{B(z)}{A(z)}.$$

Let $P_1(z), \cdots, P_r(z)$ be distinct irreducible factors of $A(z)$ in $K_0[z]$. By virtue of Lemmas 1 and 2 applied to $K_1 = K_0$, $K_2 = K$, we have then

$$\sum_{n=1}^{\infty} na_n z^{n-1} = \frac{d}{dz}\psi(z) + \sum_{j=1}^{r} \frac{R_j(z)}{P_j(z)}$$

for a rational function $\psi(z)$ in $K(z)$ and some polynomials $R_j(z)$ $(j=1, \cdots, r)$ in $K[z]$ with $\deg R_j(z) < \deg P_j(z)$ $(j=1, \cdots, r)$. We wish to show that the second term on the right-hand side of this equality is 0 (i.e. vanishes identically in z). For, otherwise, since we have for $n=1, 2, \cdots$

$$a_n = \sum_{k=1}^{m} u_{k,n}\xi_k = \frac{\sum_{k=1}^{m} u_{k,n} X_k(\sigma, \tau)}{X(\sigma)} \qquad (u_{1,n}, \cdots, u_{m,n} \in Z),$$

there would be non-zero elements $u=u(\sigma)$, $v=v(\sigma)$ in $Z[\sigma]$ such that if we write

$$u\left(\sum_{n=1}^{\infty} na_n (vz)^{n-1} - \frac{d}{d(vz)}\psi(vz)\right) = \sum_{n=1}^{\infty} nc_n z^{n-1},$$

then $c_n \in Z[\sigma, \tau]$ $(n=1, 2, \cdots)$, and, moreover, we have

$$u\sum_{j=1}^{r} \frac{R_j(vz)}{P_j(vz)} = \sum_{i=1}^{q} \frac{\alpha_i \omega_i}{1-\omega_i z} \qquad (q \geqq 1),$$

where the α_i are non-zero and algebraically integral over $Z[\sigma]$ and the ω_i are non-zero, mutually distinct, and algebraically integral over $Z[\sigma]$. It would then follow that

$$\alpha_1\omega_1^n + \alpha_2\omega_2^n + \cdots + \alpha_q\omega_q^n = nc_n \qquad (n=1, 2, \cdots).$$

We now take an arbitrary rational prime p and consider the equations

$$\alpha_1\omega_1^{jp} + \alpha_2\omega_2^{jp} + \cdots + \alpha_q\omega_q^{jp} = jpc_{jp} \qquad (j=1, 2, \cdots, q).$$

By elimination we get from these equations

$$\alpha_1 \det D = \det D_1,$$

where D is the matrix

$$(\omega_i^{jp})_{i,j=1,\cdots,q}$$

and D_1 is the one obtained from D by replacing the first column $(\omega_1^{jp})_{j=1,\cdots,q}$ by $(jpc_{jp})_{j=1,\cdots,q}$. The determinant $\det D$ is equal to $\omega_1^p \cdots \omega_q^p$ times the Vandermonde determinant $|\omega_i^{(j-1)p}|_{i,j=1,\cdots,q}$ and consequently $(\det D)^2$ is an element of $Z[\sigma]$. If we set

$$\delta(\sigma)=\omega_1\cdots\omega_q \prod_{1\leqq\mu<\nu\leqq q}(\omega_\mu-\omega_\nu),$$

then $(\delta(\sigma))^2$ is a non-zero element of $Z[\sigma]$ and

$$(\det D)^2 \equiv (\delta(\sigma))^{2p} \pmod p.$$

Let N designate the norm with respect to K/K_0 and d be the degree of α_1 over K_0. Then

$$F_1(\sigma)=(N\alpha_1)^2(\det D)^{2d}$$

is a polynomial of $Z[\sigma]$ whose coefficients are all divisible by p. Hence p must divide all the coefficients of the non-zero polynomial

$$F(\sigma)=(N\alpha_1)^2(\delta(\sigma))^{2dp}$$

in $Z[\sigma]$. However, it is apparent that this is possible only for a finite number of rational primes p, which contradicts the arbitrariness of the choice of p.

Thus we have

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = \frac{d}{dz}\psi(z)$$

and, by integration,

$$\sum_{n=0}^{\infty} a_n z^n = \psi(z)-\psi(0)+a_0,$$

concluding the proof of our theorem.

References

[1] D. G. Cantor: On arithmetic properties of coefficients of rational functions. Pacific J. of Math., **15**, 55-58 (1965).

[2] G. Pólya: Arithmetische Eigenschaften der Reihenentwicklungen rationaler Funktionen. J. Reine u. Angew. Math., **151**, 1-31 (1921).