111. *On the Rate of Growth of Blaschke Products in the Unit Circle*

By Chuji TANAKA

Mathematical Institute, Waseda University, Tokyo

(Comm. by Zyoiti SUETUNA, M.J.A., Sept. 13, 1965)

1. Introduction. Let us put

$$B(z)=\prod_{n=1}^{+\infty} b(z, a_n),$$

where $b(z, a)=|a|/a\cdot(a-z)/(1-\bar{a}z)$, $S=\sum_{n=1}^{+\infty}(1-|a_n|)<+\infty$. Then we can find the sequence $\{r_n\}$*) such that

$$(1.1)\qquad \begin{aligned}&(1)\quad 1=r_1>r_2>r_3\cdots\cdots\to 0,\\ &(2)\quad \sum_{n=1}^{+\infty} 1/r_n^2\cdot(1-|a_n|)<+\infty.\end{aligned}$$

For the sake of convenience, we introduce some notations:

(1) $D(e^{i\varphi}, \vartheta)=\{z;\ |\arg(1-ze^{-i\varphi})|\leqq\vartheta<\pi/2,\ |z-e^{i\varphi}|\leqq\cos\vartheta\}$.

(2) $D(e^{i\varphi}, r_1, r_2)=(|z-r_1e^{i\varphi}|\leqq 1-r_1)\cap(|z-r_2e^{i\varphi}|\geqq 1-r_2)$, $(0<r_1<r_2<1)$.

(3) $\mathscr{D}=\bigcap_n\{z;\ \rho(z, a_n)\geqq R_n\}$, where $\rho(a, b)$: The non-Euclidean hyperbolic distance between a and b, $R_n=\tanh^{-1} r_n$ $(n=1, 2, \cdots)$.

(4) $S(\varepsilon)=\sum_{|a_n-e^{i\varphi}|<\varepsilon} 1/r_n^2\cdot(1-|a_n|)$.

Then we can state our theorems as follows:

Theorem 1.

$$(1.2)\qquad \lim_{\substack{|z|\to 1\\ z\in\mathscr{D}}}(1-|z|)\log|1/B(z)|=0.$$

As its immediate consequences, we get following:

Corollary 1.

(1.3) $\lim|z-e^{i\varphi}|\cdot\log|1/B(z)|=0$ *uniformly as* $z\to e^{i\varphi}$ *inside* $D(e^{i\varphi}, \vartheta)\cap\mathscr{D}$.

Corollary 2. *If there exists no* $\{a_n\}$ *in the sector* S: $\alpha\leqq\arg(1-ze^{-i\varphi})\leqq\beta(-\pi/2<\alpha<\beta<\pi/2)$, *then* $\lim|z-e^{i\varphi}|\cdot\log|1/B(z)|=0$ *uniformly as* $z\to e^{i\varphi}$ *inside the subsector of* S.

As an interesting application of Corollary 2, we can establish

Theorem 2. *If the sequence* $\{a_n\}$ *lies on the chord* $L: \arg(1-ze^{-i\varphi})=\vartheta$ $(|\vartheta|<\pi/2)$, *then* L *is Julia- line for* $f(z)=B(z)\cdot\exp\{\alpha\cdot(e^{i\varphi}+z)/(e^{i\varphi}-z)\}$ $(\alpha>0)$.

Under additional conditions, we can prove more precise theorems than Theorem 1:

Theorem 3. *If* $\overline{\lim}_{\varepsilon\to+0} S(\varepsilon)/\varepsilon^2<+\infty$, *then* $\underline{\lim}|B(z)|>0$ *as* $z\to e^{i\varphi}$ *inside* $\mathscr{D}\cap(D(e^{i\varphi}, \vartheta)\cup D(e^{i\varphi}, r_1, r_2))$.

---

*) Vide lemma 1.

Theorem 4. *If* $\lim_{\varepsilon\to+0} S(\varepsilon)/\varepsilon^{\alpha}=0$ $(\alpha>2)$, *then* $\lim|B(z)|=1$ *as* $z\to e^{i\varphi}$ *inside* $\mathcal{D}\cap(\mathcal{D}(e^{i\varphi},\vartheta)\cup D(e^{i\varphi},r_1,r_2))$.

2. Lemmas. To prove these theorems, we need some lemmas:

Lemma 1. *There exists the sequence* $\{r_n\}$ *satisfying* (1.1).

Proof. Dini's theorem ([1] p. 293) states that, if $\sum_{n=1}^{+\infty} c_n$ is a convergent series of positive terms, then $\sum_{n=1}^{+\infty} c_n/(c_n+c_{n+1}\cdots)^{\alpha}$ converges when $\alpha<1$, and diverges when $\alpha\geqq 1$. If we put

$$c_n=1-|a_n|,\quad r_n=\Bigl(\sum_{k=n}^{+\infty} c_k/\sum_{k=1}^{+\infty} c_k\Bigr)^{\frac{\alpha}{2}}\quad (0<\alpha<1),$$

then lemma 1 follows immediately from Dini's theorem.

Lemma 2.

$$\log|(1-\bar{a}z)/(z-a)|<2(1-|a|)(1-|z|)/|z-a|^2 \tag{2.1}$$

for $|a|<1$, $|z|<1$.

Proof. By the inequality: $\log(1+x)<x$ for $x>0$, we have

$$\begin{aligned}\log|(1-\bar{a}z)/(z-a)|&=1/2\log\{1+(1-|a|^2)(1-|z|^2)/|z-a|^2\}\\&<1/2\cdot(1-|a|^2)(1-|z|^2)/|z-a|^2\\&<2(1-|a|)(1-|z|)/|z-a|^2,\end{aligned}$$

which proves Lemma 2.

Lemma 3. *Put* $\rho=|z-e^{i\varphi}|<\varepsilon$.

(1) *If* $z\in D(e^{i\varphi},\vartheta)\cap\mathcal{D}$, $\overline{\lim}_{\varepsilon\to+0}\rho/\varepsilon=\Delta<1$, *then*

$$\log|1/B(z)|<4S/(1-\Delta)^2\cdot\rho/\varepsilon^2+4/\cos\vartheta\cdot S(\varepsilon)/\rho \tag{2.2}$$

for sufficiently small ε.

(2) *If* $z\in D(e^{i\varphi},r_1,r_2)\cap\mathcal{D}$, $\overline{\lim}_{\varepsilon\to+0}\rho/\varepsilon=\Delta<1$, *then*

$$\log|1/B(z)|<2S/(1-\Delta)^2\cdot r_2/(1-r_2)\cdot\rho^2/\varepsilon^2+4(1-r_1)/r_1\cdot S(\varepsilon)/\rho^2 \tag{2.3}$$

for sufficiently small ε.

Proof. If $z\in\mathcal{D}$, then $\rho(z,a_n)\geqq\tanh^{-1}r_n$ $(n\geqq1)$, i.e. $|(z-a_n)/(1-\bar{a}_nz)|\geqq r_n$ $(n\geqq1)$, so that

$$|z-a_n|/(1-|z|)>r_n\quad(n\geqq1). \tag{2.4}$$

Hence, by (2.1)

$$S_1=\sum_{\left(\substack{|a_n-e^{i\varphi}|<\varepsilon\\ z\in\mathcal{D}}\right)}\log|(1-\bar{a}_nz)/(z-a_n)|<2S(\varepsilon)\cdot1/(1-|z|). \tag{2.5}$$

Similarly

$$\begin{aligned}S_2=&\sum_{\left(\substack{|a_n-e^{i\varphi}|\geqq\varepsilon,\\ |z-e^{i\varphi}|=\rho<\varepsilon}\right)}\log|(1-\bar{a}_nz)/(z-a_n)|\\&<2\sum_{|a_n-e^{i\varphi}|\geqq\varepsilon}(1-|a_n|)\cdot(1-|z|)/(\varepsilon-\rho)^2\\&<2S\cdot(1-|z|)/(\varepsilon-\rho)^2.\end{aligned} \tag{2.6}$$

Putting $z=(1-\rho e^{i\theta})e^{i\varphi}$, we have easily

$$1-|z|^2=\rho(2\cos\theta-\rho). \tag{2.7}$$

By (2.5), (2.6), and (2.7)

$$\log_{\left(\substack{|z-e^{i\varphi}|=\rho<\varepsilon\\ z\in\mathcal{D}}\right)}|1/B(z)|<2S/(1-\Delta)^2\cdot\rho(2\cos\theta-\rho)/\varepsilon^2+4S(\varepsilon)/\rho(2\cos\theta-\rho) \tag{2.8}$$

for sufficiently small ε.

If $z\in D(e^{i\varphi},\vartheta)\cap\mathcal{D}$, then $2\cos\theta-\rho\geqq\cos\vartheta$, so that by (2.8)

$$\log_{\substack{|z-e^{i\varphi}|=\rho<\varepsilon\\ z\in D(e^{i\varphi},\vartheta)\cap\mathcal{D}}}|1/B(z)|<4S/(1-\Delta)^2\cdot\rho/\varepsilon^2+4/\cos\vartheta\cdot S(\varepsilon)/\rho$$

for sufficiently small ε, which proves (2.2).

If $z\in D(e^{i\varphi},r_1,r_2)$, then by simple computations

$$\rho^2\cdot r_1/(1-r_1)\leqq\rho(2\cos\theta-\rho)\leqq\rho^2\cdot r_2/(1-r_2),$$

so that by (2.8)

$$\log_{\substack{|z-e^{i\varphi}|=\rho<\varepsilon\\ z\in D(e^{i\varphi},r_1,r_2)\cap\mathcal{D}}}|1/B(z)|<2S/(1-\Delta)^2\cdot r_2/(1-r_2)\cdot\rho^2/\varepsilon^2+4(1-r_1)/r_1\cdot S(\varepsilon)/\rho^2$$

for sufficiently small ε, which proves (2.3)

3. Proofs of Theorems 1–4.

Proof of Theorem 1. By (2.1) and (2.4), if $z\in\mathcal{D}$,

$$(3.1)\qquad (1-|z|)\log|1/B(z)|<2\sum_{n=1}^{+\infty}(1-|a_n|)((1-|z|)/|z-a_n|)^2$$
$$<2\left\{\sum_{n=1}^{N}(1-|a_n|)((1-|z|)/|z-a_n|)^2+\sum_{n=N+1}^{+\infty}1/r_n^2\cdot(1-|a_n|)\right\},$$

N being any fixed integer. For any given $\varepsilon>0$, there exists $N(\varepsilon)$ such that $\sum_{n=N+1}^{+\infty}1/r_n^2\cdot(1-|a_n|)<\varepsilon$ for $N\geqq N(\varepsilon)$. Hence, by (3.1)

$$(1-|z|)\log|1/B(z)|<2\left\{\sum_{n=1}^{N}(1-|a_n|)((1-|z|)/|z-a_n|)^2+\varepsilon\right\}$$

for $N\geqq N(\varepsilon)$, so that

$$0\leqq\overline{\lim_{\substack{|z|\to1\\ z\in\mathcal{D}}}}(1-|z|)\log|1/B(z)|\leqq2\varepsilon.$$

Letting $\varepsilon\to+0$, we have $\lim_{\substack{|z|\to1\\ z\in\mathcal{D}}}(1-|z|)\log|1/B(z)|=0$, which proves proves Theorem 1.

Since $\cos\vartheta/2\leqq(1-|z|)/|z-e^{i\varphi}|$ for $z\in D(e^{i\varphi},\vartheta)$,

$$0\leqq|z-e^{i\varphi}|\cdot\log|1/B(z)|\leqq2\sec\vartheta\cdot(1-|z|)\cdot\log|1/B(z)|$$

for $z\in D(e^{i\varphi},\vartheta)$. Hence, Corollary 1 follows immediately from Theorem 1.

If there exists no $\{a_n\}$ in the sector $S:-\pi/2<\alpha\leqq\arg(1-ze^{-i\varphi})\leqq\beta<+\pi/2$, then by (1.1) (1), the hyperbolic disks: $\rho(z,a_n)\leqq R_n$ $n\geqq N$ (N: sufficiently large integer) are not contained in the fixed subsector of S. Therefore Corollary 2 is an immediate consequence of Corollary 1.

Proof of Theorem 2. Without any loss of generality, we can assume that $\varphi=0$. By Corollary 2, $\lim|z-1|\cdot\log|B(z)|=0$ as $z\to1$ on the chords: $\arg(1-z)=\vartheta\pm\varepsilon$, ε being any positive constant such that $-\pi/2<\vartheta-\varepsilon<\vartheta+\varepsilon<+\pi/2$. Hence, for any $\delta>0$,

$$(3.2)\qquad |B(z)|>\exp\{-\delta/(1-|z|)\}$$

for $\arg(1-z)=\vartheta\pm\varepsilon$, $|1-z|\leqq\Delta(\delta)$, $\Delta(\delta)$ being a constant dependent upon δ. By simple computation,

$$(1-|z|)/|1-z| \geqq 1/2 \cdot \min\{\cos(\vartheta+\varepsilon), \cos(\vartheta-\varepsilon)\} = \delta^*/2$$

for $\arg(1-z)=\vartheta\pm\varepsilon$, $|1-z|\leqq\delta^*$, so that

(3.3) $$\mathscr{R}((1+z)/(1-z)) = (1-|z|^2)/|1-z|^2 > (\delta^*/2)^2 \cdot 1/(1-|z|).$$

By (3.2) and (3.3)

$$|f(z)| > \exp\{1/(1-|z|)\cdot(\alpha\cdot(\delta^*/2)^2-\delta)\}$$

for $\arg(1-z)=\vartheta\pm\varepsilon$, $|1-z|\leqq\min(\Delta(\delta), \delta^*)$. Taking δ so small that $\alpha(\delta^*/2)^2>\delta$, $w=f(z)$ tends to ∞ as $z\to 1$ along the chords : $\arg(1-z)=\vartheta\pm\varepsilon$. Since $f(a_n)=0$ $n\geqq 1$, by Gross-Iversen's theorem ([2] p. 5) the cluster set of $w=f(z)$ at $z=1$ inside the sector $S: |\arg(1-z)-\vartheta|\leqq\varepsilon$ is the whole w-plane and $f(z)$ takes every finite value, except perhaps one, infinitely many times in S. Since ε is arbitrary, the chord L: $\arg(1-z)=\vartheta$ is Julia-line, which proves Theorem 2.

Proof of Theorem 3. If $z\in D(e^{i\varphi}, \vartheta)\cap\mathscr{D}$, then by Lemma 3 (1), in which we put $\rho=\varepsilon^2$, $\Delta=0$, we have

(3.4) $$\overline{\lim}\log|1/B(z)| \leqq 4S + 4/\cos\vartheta\cdot\overline{\lim_{\varepsilon\to+0}} S(\varepsilon)/\varepsilon^2 < +\infty$$

as $z\longrightarrow e^{i\varphi}$ inside $D(e^{i\varphi}, \vartheta)\cap\mathscr{D}$.

If $z\in D(e^{i\varphi}, r_1, r_2)\cap\mathscr{D}$, then by lemma 3 (2), in which we put $\rho=\varepsilon/2$, $\Delta=1/2$, we get

(3.5) $$\overline{\lim}\log|1/B(z)| \leqq 2Sr_2/(1-r_2) + 16(1-r_1)/r_1\cdot\overline{\lim_{\varepsilon\to+0}} S(\varepsilon)/\varepsilon^2 < +\infty$$

as $z\longrightarrow e^{i\varphi}$ inside $D(e^{i\varphi}, r_1, r_2)\cap\mathscr{D}$. By (3.4) and (3.5), Theorem 3 is completely established.

Proof of Theorem 4. If $z\in D(e^{i\varphi}, \vartheta)\cap\mathscr{D}$, then by Lemma 3 (1), in which we put $\rho=\varepsilon^\alpha$, $\Delta=0$, we obtain

$$0\leqq\overline{\lim}\log|1/B(z)|\leqq 0, \text{ i.e. } \lim|B(z)|=1$$

as $z\longrightarrow e^{i\varphi}$ inside $D(e^{i\varphi}, \vartheta)\cap\mathscr{D}$.

If $z\in D(e^{i\varphi}, r_1, r_2)\cap\mathscr{D}$, then by Lemma 3 (2), in which we put $\rho=\varepsilon^{\frac{\alpha}{2}}$, $\Delta=0$, we get

$$0\leqq\overline{\lim}\log|1/B(z)|\leqq 0, \text{ i.e. } \lim|B(z)|=1$$

as $z\longrightarrow e^{i\varphi}$ inside $D(e^{i\varphi}, r_1, r_2)\cap\mathscr{D}$. Thus our theorem is completely proved.

References

[1] K. Knopp: Theory and Application of Infinite Series. London and Glasgow (1928).
[2] K. Noshiro: Cluster Sets. Springer, Berlin (1960).