154. *Algebraic Aspects of Non-Self Adjoint Operators*[*)]

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The purpose of this paper is to present the algebraic approach to the theory of non-self adjoint operators on Hilbert space by means of the theory of von Neumann algebras. We know that this approach was quite adequate for the general treatment of normal operators on Hilbert space with the aid of the pleasant features of abelian von Neumann algebras. More generally, our approach with its technical advantages will serve to describe the structure of non-normal operators. In the present paper, we shall introduce a new class of operators on Hilbert space. An operator A is said to be *primary* if the von Neumann algebra $R(A)$ generated by A is a factor (i.e., the center of $R(A)$ is the scalar multiples of the identity operator). Then it may be considered that the spectral decomposition of a normal operator A is essentially nothing but the decomposition of A into primary normal operators (which are scalar operators). Moreover, we know that an isometry is decomposed as the direct sum of a unitary operators and a unilateral shift. As we have shown in [7] (cf. [3; Theorem 1]), a unilateral shift is a primary operator. From this fact we can easily see that a non-scalar isometry is a unilateral shift if and only if it is primary. Therefore, with the aid of the spectral theorem for a unitary operator, the above decomposition of an isometry V essentially is the decomposition of V into primary isometric operators. From this point of view, the decomposition of an operator into primary operators may be regarded as a kind of spectral decomposition.

We shall concern ourselves with the class of operators whose imaginary part are completely continuous. M. S. Brodskiĭ-M. S. Livšic [2] and M. S. Livšic [5] have developed the theory of subdiagonalization for operators whose imaginary parts belong to the trace class. Our purpose is to establish the decomposition of an operator of this class into primary operators belonging to this same class. Consequently, we shall be able to see some algebraic aspects of operators of this class.

For the sake of simplicity, we shall assume that our Hiblert

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space is separable. By an operator we shall always understand a bounded linear transformation on a Hilbert space. By a von Neumann algebra we shall mean a self-adjoint operator algebra with the identity operator which is closed in the weak topology. For the basic definitions and notations concerning the theory of von Neumann algebras, we shall refer to the book of J. Dixmier [1].

1. Let A be an operator on a Hilbert space H whose imaginary part $Im(A)=\frac{1}{2i}(A-A^*)$ is completely continuous. Our object is to decompose essentially the operator A into primary operators. We shall denote by K the range of $Im(A)$, i.e.,

$$K=\frac{1}{2i}(A-A^*)H$$

and the projection on the subspace K will be denoted by E. Moreover we shall consider the subspace H_1 generated by vectors of the form $A^n\varphi(\varphi\in K,\ n=0,1,2,\cdots)$ and denote by P the projection on H_1. Since $Im(A)$ is a self-adjoint completely continuous operator, it is well known that there exists an orthonormal basis in H whose elements are proper values of $Im(A)$. Therefore, if we denote by $\{\mu_k\}(k\in N)$ the countable family of all distinct non-zero proper values of $Im(A)$ and by E_k the projection on the proper subspace corresponding to μ_k, each proper subspace E_kH is finite dimensional and $E=\sum_{k\in N}E_k$. The key observation in our decomposition is that each $E_k(k\in N)$ and E are projections in $R(A)$. Then we can see that the projection P is the central support of E (that is, the minimal central projection containing E). From this fact it follows that A is decomposed into the form

$$A=A_{I-P}\oplus A_P$$

where A_P (resp. A_{I-P}) is the restriction of A on PH (resp. $(I-P)H$) and A_{I-P} is a self-adjoint operator. Keeping in mind that each projection E_k in $R(A)$ is finite dimensional, we can choose a family of minimal projections $F_i(i\in I)$ in $R(A)$ such that the central supports P_i of F_i are mutually orthogonal and $P=\sum_{i\in I}P_i$. Here, by making use of the standard technique on von Neumann algebras, we can decompose A_P by the family of central projections $P_i(i\in I)$ into primary operators.

THEOREM 1. *An operator A with completely continuous imaginary part on a Hilbert space H is decomposed by a unique countable family of mutually orthogonal central projections $P_0P_i(i\in I)$ in $R(A)$ into the form*

$$A=A_{P_0}\oplus\sum_{i\in I}\oplus A_{P_i}$$

where the restriction A_{P_0} of A to P_0H is a self-adjoint operator, the restriction A_{P_i} of A to $P_iH(i\in I)$ is a primary operator with com-

pletely continuous imaginary part and $P=\sum_{i\in I}P_i$ *is the projection on the subspace generated by vectors of the form* $A^n\varphi(\varphi\in Im(A)H, n=0,1,2,\cdots)$.

Certainly the essentials of our result is the decomposition of $R(A)$ into factors in the reduction theory of von Neumann [6], but it should be noticed that the character of the operator A has induced a more simple decomposition of $R(A)$. What our theorem means is quite well illustrated by taking a normal operator of this class.

COROLLARY 1. *Let* A *be a normal operator with completely continuous imaginary part. Then* A *is uniquely expressed by a countable family of mutually orthogonal projections* P_0, $P_i(i\in I)$ *in* $R(A)$ *as follows*:

$$A=AP_0+\sum_{i\in I}\lambda_i P_i$$

where AP_0 *is a self-adjoint operator, each projection* $P_i(i\in I)$ *is finite dimensional,* $I=P_0+\sum_{i\in I}P_i$ *and* $\{\lambda_i\}$ $(i\in I)$ *is the family of non-real proper values of* A.

Here, we shall mention a very important special class of our operators, that is, the class of operators whose imaginary parts are finite dimensional operators. Let A be an operator with finite dimensional imaginary part. Then the dimension r of the range of $Im(A)$ is called the *non-hermitian rank* of A. In this case, the main part of Theorem 1 may be stated as follows.

COROLLARY 2. *An operator* A *with non-hermitian rank* r *is decomposed by a unique family of mutually orthogonal central projections* $P_0, P_1, \cdots, P_n$ *in* $R(A)$ *into the form*

$$A=A_{P_0}\oplus A_{P_1}\oplus\cdots\oplus A_{P_n}$$

where A_{P_0} *is a self-adjoint operator,* $A_{P_i}(i=1,2,\cdots,n)$ *is a primary operator with non-hermitian rank* k_i *and* $\sum_{i=1}^{n}k_i=r$.

2. The algebraic structure of an operator A is closely related to the type of the von Neumann algebra $R(A)$. We say that an operator A is *of type I* if $R(A)$ is of type I, and furthermore we say that a primary operator A is *of type* I_n (resp. *type* I_∞) if the factor $R(A)$ is of type I_n (resp. type I_∞). Then a natural question coming to our mind is this: which non-normal operators are of type I? A few answers to this question are known. We know that an isometry is of type I ([7]). Now we can add a satisfactory answer to this question. Indeed, the determination of the type of operators with completely continuous imaginary part is visible from Theorem 1.

THEOREM 2. *An operator* A *with completely continuous imaginary part is of type I.*

For the proof, we may observe that each operator A_{P_i} in Theorem 1 generates the von Neumann algebra which is spatial isomorphic to

$C \otimes \mathcal{L}(\mathfrak{H})$, where C is the scalar multiples of the identity operator on $F_i H$ and $\mathcal{L}(\mathfrak{H})$ is the algebra of all operators on a finite or infinite dimensional Hilbert space $\mathfrak{H}$. This class of operators contains several important classes of operators which appear in many fields of analysis. In particular, we should notice the following fact ([8]).

COROLLARY. *A completely continuous operator is of type I.*

Naturally, a non-scalar primary operator A with completely continuous imaginary part admits to be of type I_∞, but the commutant of $R(A)$ is necessarily of type $I_n (n=1, 2, \cdots)$. It seems that the algebraic aspect of primary operators of our class is well expounded by this fact. Actually, from this fact we can deduce the algebraic structure of a primary operator with finite non-hermitian rank in a simple way. The details of this subject will appear elsewhere with complete proofs of our theorems.

3. M. S. Brodskiĭ-M. S. Livšic [2] have established the basic properties of the spectrum of an operator with completely continuous imaginary part. For example, every non-real point of the spectrum of the operator is a proper value and its proper subspace is finite dimensional. Here is a significant and attractive problem: how does the algebraic simplicity of a primary operator effect its spectrum? Although many questions about it are left to be settled in the future, we shall have some comments on the spectrum of primary operators as a step toward our problem.

(A) *Every point of the spectrum of a non-scalar primary operator A with completely continuous imaginary part lies in the open disc $D_A = \{\lambda : |\lambda| < \|A\|\}$.*

Indeed, every proper value of a non-scalar primary operator A lies in the open disc D_A.

(B) *A primary operator with non-hermitian rank* 1 *does not have a real proper value.*

From this result we can see that Theorem 1 yields the spectral decomposition of an operator A with non-hermitian rank 1 whose spectrum is real, in the sense that A is decomposed by a central projection in $R(A)$ into the form $A = B \oplus C$ where B (resp. C) has a pure point (resp. continuous) spectrum. Among the primary operators with non-hermitian rank 1, a quasi-nilpotent operator is completely determined as follows.

(C) *A quasi-nilpotent primary operator with non-hermitian rank* 1 *is unitarily equivalent to the integral operator λV on $L_2[0, 1]$, where λ is a scalar and V is the operator defined by*

$$(Vf)(x) = i \int_0^x f(t)dt .$$

This is the slight modification of the notable result obtained in [2] and [4].

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