153. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XVII

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Let $T(\lambda)$ be the function treated in Theorems 43, 44, and 45 of the preceding paper. Namely $T(\lambda)$ has as its singularity every point belonging to the bounded set $\overline{\{\lambda_\nu\}} \cup \left[\bigcup_{j=1}^{n} D_j\right]$ where the denumerably infinite set $\{\lambda_\nu\}$ is everywhere dense on a closed or an open rectifiable Jordan curve Γ and satisfies the condition that for any small positive ε the circle $|\lambda| = \sup_\nu |\lambda_\nu| + \varepsilon$ contains the mutually disjoint closed sets $\Gamma, D_1, D_2, \cdots, D_{n-1}$, and D_n inside itself [cf. Proc. Japan Acad., 40 (7), 492–497 (1964)]. In this paper we are mainly concerned with the distribution of c-points of the sum of the first and second principal parts of $T(\lambda)$ in the domain $\{\lambda: \sup_\nu |\lambda_\nu| < |\lambda| < \infty\}$, on the assumption that c is an arbitrary finite complex number.

Theorem 46. Let $\chi(\lambda)$ be the sum of the first and second principal parts of the above-mentioned function $T(\lambda)$; let $\sigma = \sup_\nu |\lambda_\nu|$; let c be an arbitrarily given finite non-zero complex number; let $n(\rho, c)$, $(\sigma < \rho < \infty)$, be the number of all the c-points, with due count of multiplicity, of $\chi(\lambda)$ in the domain $\Delta_\rho\{\lambda: \rho < |\lambda| < \infty\}$; let

$$N(\rho, c) = \int_\rho^\infty \frac{n(r, c)}{r} dr \qquad (\sigma < \rho < \infty);$$

and let

$$m(\rho, c) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{[\chi(\rho e^{-it}), c]} dt \qquad (\sigma < \rho \leqq \infty),$$

where

$$[\chi(\rho e^{-it}), c] = \frac{|\chi(\rho e^{-it}) - c|}{\sqrt{(1 + |\chi(\rho e^{-it})|^2)(1 + |c|^2)}}.$$

Then the equality

$$N(\rho, c) + m(\rho, c) - m(\infty, c) = \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{1 + |\chi(\rho e^{-it})|^2}\, dt$$

holds for every ρ with $\sigma < \rho < \infty$; and in addition, $N(\rho, c)$, $m(\rho, c) - m(\infty, c)$, and the right-hand definite integral tend to 0 as ρ becomes infinite.

Proof. If we now consider the function $f(\lambda) \equiv \chi\left(\frac{\rho^2}{\lambda}\right)$, $(\sigma < \rho < \infty)$, of a complex variable λ, then $f(\lambda)$ is regular in the domain $D\{\lambda: 0 \leqq$

$|\lambda| \leqq \rho\}$ because of the fact that to the domain D of f there corresponds the domain $D'\left\{\frac{\rho^2}{\lambda}: \rho \leqq \frac{\rho^2}{|\lambda|} \leqq \infty\right\}$ of χ and that as will be seen from the definition of $\chi(\lambda)$, $f(0)=\chi(\infty)=0$. If we next denote all the c-points, $(c\neq 0, \infty)$, of $f(\lambda)$ in the domain $\mathfrak{D}_\rho\{\lambda: 0<|\lambda|<\rho\}$ by $a_1, a_2, \cdots, a_{n(\rho)}$, c-points of orders higher than 1 being accounted for by listing the corresponding points a_κ an appropriate number of times, then all the c-points of $\chi(\lambda)$ in the domain $\Delta_\rho\{\lambda: \rho<|\lambda|<\infty\}$ are given by $\frac{\rho^2}{a_1}\equiv b_1, \frac{\rho^2}{a_2}\equiv b_2, \cdots, \frac{\rho^2}{a_{n(\rho)}}\equiv b_{n(\rho)}$ (repeated according to the respective orders) and the application of Jensen's theorem to $f(\lambda)-c$ yields the equality

$$\log|f(0)-c| + \log\frac{\rho^{n(\rho)}}{|a_1 a_2 \cdots a_{n(\rho)}|} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(\rho e^{it})-c|\,dt$$
$$(\sigma<\rho<\infty).$$

For convenience' sake, we shall here rewrite the number $n(\rho)$ of all the c-points, with due count of multiplicity, of $f(\lambda)$ in $\mathfrak{D}_\rho$ by $n_1(\rho, c)$. Then $n_1(\rho, c)$ equals $n(\rho, c)$ defined in the statement of the present theorem and it can be verified by direct computation that

$$\int_0^\rho \frac{n_1(r, c)}{r}dr = \log\frac{\rho^{n(\rho)}}{|a_1 a_2 \cdots a_{n(\rho)}|},$$
$$\int_\rho^\infty \frac{n(r, c)}{r}dr = \log\frac{|b_1 b_2 \cdots b_{n(\rho)}|}{\rho^{n(\rho)}},$$

and hence

$$\int_0^\rho \frac{n_1(r, c)}{r}dr = \int_\rho^\infty \frac{n(r, c)}{r}dr = N(\rho, c).$$

Since, moreover, $f(0)=\chi(\infty)=0$, the equality establised before is rewritten in the form

$$\log|\chi(\infty)-c| + N(\rho, c) = \frac{1}{2\pi}\int_0^{2\pi}\log|\chi(\rho e^{-it})-c|\,dt \qquad (\sigma<\rho<\infty).$$

Remembering the definition concerning $[\chi(\rho e^{-it}), c]$ and subtracting $\frac{1}{2\pi}\int_0^{2\pi}\log\sqrt{1+|\chi(\rho e^{-it})|^2}\,dt + \log\sqrt{1+|c|^2}$ from the left and right sides of the final equality, we have

$$-\log\frac{1}{[\chi(\infty), c]} - \frac{1}{2\pi}\int_0^{2\pi}\log\sqrt{1+|\chi(\rho e^{-it})|^2}\,dt + N(\rho, c)$$
$$= -\frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{[\chi(\rho e^{-it}), c]}dt,$$

where it is easily verified that

$$m(\infty, c) = \log\frac{1}{[\chi(\infty), c]} = \log\sqrt{1+|c|^{-2}}.$$

Consequently we obtain the desired relation

$$N(\rho, c)+m(\rho, c)-m(\infty, c)=\frac{1}{2\pi}\int_0^{2\pi}\log\sqrt{1+|\chi(\rho e^{-it})|^2}\,dt,$$

where it is easily found that each of $N(\rho, c)$, $m(\rho, c)-m(\infty, c)$, and the definite integral concerning $\log\sqrt{1+|\chi(\rho e^{-it})|^2}$ tends to 0 as ρ becomes infinite.

Thus the present theorem has been proved.

Remark 1. Let $M_f(\rho)$ and $M_\chi(\rho)$ denote the maximum moduli of $f(\lambda)$ and $\chi(\lambda)$ on the circle $|\lambda|=\rho$ respectively. In Theorem 43 we have already proved that $M_\chi(\rho')\leqq M_\chi(\rho)$ for every pair of ρ', ρ with $\sigma<\rho<\rho'<\infty$. Since, on the other hand, the function $f(\lambda)$ defined at the beginning of the proof of Theorem 46 is regular in the domain $\{\lambda: 0\leqq|\lambda|\leqq\rho\}$ with $\sigma<\rho<\infty$, $M_f(\rho')<M_f(\rho'')$ for every pair of ρ', ρ'' with $0<\rho'<\rho''\leqq\rho$. Hence $M_\chi(\rho')<M_\chi(\rho)$ for every pair of ρ', ρ with $\sigma<\rho<\rho'<\infty$.

Remark 2. The result of Theorem 46 corresponds to Nevanlinna's first fundamental theorem concerning a meromorphic function and shows that $N(\rho, c)+m(\rho, c)-m(\infty, c)$ corresponds to a modified form of the characteristic function due to Ahlfors and Shimizu.

Theorem 47. Let $T(\lambda)$, $\chi(\lambda)$, and σ be the same notations as before; let C be the positively oriented circle $|\lambda|=\rho$ with $\sigma<\rho<\infty$; let

$$\frac{1}{2\pi i}\int_C\frac{T(\lambda)}{\lambda^{-\mu+1}}d\lambda=0\quad(\mu=1, 2, \cdots, k-1),$$

$$C_{-k}=\frac{1}{2\pi i}\int_C\frac{T(\lambda)}{\lambda^{-k+1}}d\lambda\neq 0;$$

let $\tilde{n}(\rho, 0)$, $(\sigma<\rho<\infty)$, be the number of all the zeros, with due count of multiplicity, of $\chi(\lambda)$ in the domain $\{\lambda: \rho<|\lambda|<\infty\}$; let

$$\tilde{N}(\rho, 0)=\int_\rho^\infty\frac{\tilde{n}(r, 0)}{r}dr\qquad(\sigma<\rho<\infty);$$

and let

$$\tilde{m}(\rho, 0)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{|C_{-k}|}{[\chi(\rho e^{-it}), 0]\rho^k}dt\qquad(\sigma<\rho<\infty).$$

Then

$$\tilde{N}(\rho, 0)+\tilde{m}(\rho, 0)=\frac{1}{2\pi}\int_0^{2\pi}\log\sqrt{1+|\chi(\rho e^{-it})|^2}dt\qquad(\sigma<\rho<\infty),$$

where $\tilde{N}(\rho, 0)$, $\tilde{m}(\rho, 0)$, and the right-hand member tend to 0 as ρ becomes infinite.

Proof. Since, as already shown, $\chi(\lambda)=\sum_{\mu=1}^{\infty}\frac{C_{-\mu}}{\lambda^\mu}$, $(\sigma<|\lambda|)$, where $C_{-\mu}=\frac{1}{2\pi i}\int_C\frac{T(\lambda)}{\lambda^{-\mu+1}}d\lambda$ [cf. Proc. Japan Acad., 40 (8), 654–659 (1964)],

the function
$$F(\lambda)=\begin{cases}\dfrac{\chi\left(\dfrac{\rho^2}{\lambda}\right)}{\lambda^k} & (0<|\lambda|\leqq\rho,\ \sigma<\rho<\infty)\\ \dfrac{C_{-k}}{\rho^{2k}} & (\lambda=0)\end{cases}$$
can be expressed in the form $F(\lambda)=\sum_{\mu=k}^{\infty}\dfrac{C_{-\mu}\lambda^{\mu-k}}{\rho^{2\mu}}$ by virtue of the hypothesis on $C_{-\mu}$, $(\mu=1, 2, \cdots, k)$, and is regular in the domain $\{\lambda: 0\leqq|\lambda|\leqq\rho\}$. Let now all the zeros of $F(\lambda)$ in the domain $\{\lambda: 0<|\lambda|<\rho\}$ be denoted by $\alpha_1, \alpha_2, \cdots, \alpha_{n(\rho)}$, zeros of orders higher than 1 being accounted for by listing the corresponding points α_κ an appropriate number of times. Then all the zeros of $\chi(\lambda)$ in the domain $\{\lambda: \rho<|\lambda|<\infty\}$ are given by $\dfrac{\rho^2}{\alpha_1}, \dfrac{\rho^2}{\alpha_2}, \cdots, \dfrac{\rho^2}{\alpha_{n(\rho)}}$ (repeated according to the respective orders) and yet the equality
$$\log|F(0)|+\log\frac{\rho^{n(\rho)}}{|\alpha_1\alpha_2\cdots\alpha_{n(\rho)}|}=\frac{1}{2\pi}\int_0^{2\pi}\log|F(\rho e^{it})|\,dt \qquad (\sigma<\rho<\infty)$$
holds in accordance with Jensen's theorem. Furthermore this equality is rewritten in the form
$$\log|C_{-k}|-\log\rho^k+\log\frac{\rho^{n(\rho)}}{|\alpha_1\alpha_2\cdots\alpha_{n(\rho)}|}=\frac{1}{2\pi}\int_0^{2\pi}\log|\chi(\rho e^{-it})|\,dt \qquad (\sigma<\rho<\infty).$$

On the other hand, by reasoning exactly like that used in the course of the proof of the preceding theorem it is verified that
$$\log\frac{\rho^{n(\rho)}}{|\alpha_1\alpha_2\cdots\alpha_{n(\rho)}|}=\tilde{N}(\rho, 0)$$
and hence that
$$\tilde{N}(\rho, 0)+\tilde{m}(\rho, 0)=\frac{1}{2\pi}\int_0^{2\pi}\log\sqrt{1+|\chi(\rho e^{-it})|^2}\,dt. \qquad (\sigma<\rho<\infty),$$
where
$$\tilde{m}(\rho, 0)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{|C_{-k}|}{[\chi(\rho e^{-it}), 0]\rho^k}dt.$$

In addition, it is easily found that each of $\tilde{N}(\rho, 0)$, $\tilde{m}(\rho, 0)$ and the above definite integral concerning $\log\sqrt{1+|\chi(\rho e^{-it})|^2}$ tends to 0 as ρ becomes infinite.

The proof of the theorem has thus been finished.