151. Commuting Dilations of Self-adjoint Operators

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All operators considered are bounded self-adjoint. Given an operator A on the Hilbert space $\mathfrak{H}$, an operator T on a superspace $\supseteq \mathfrak{H}$ is called a *dilation* of A in case $Af = PTf$ for $f \in \mathfrak{H}$, where P is the projection onto $\mathfrak{H}$. A family $\mathfrak{A}$ of operators on $\mathfrak{H}$ is said to be of $\langle \alpha, \beta \rangle$ *type* in case there is a commutative family $\mathfrak{B}$ of operators on a superspace such that spectra of every member of $\mathfrak{B}$ are contained in the closed interval $[\alpha, \beta]$ and every member of $\mathfrak{A}$ finds a dilation in $\mathfrak{B}$. In the above definition the superspace is not fixed throughout, but depends on $\mathfrak{A}$. In this note an intrinsic description of being of $\langle \alpha, \beta \rangle$ type is given and some of related problems are discussed.

Since under a homothety $A \rightarrow \rho A + \xi I$, I being the identity operator, the dilation type $\langle \alpha, \beta \rangle$ changes to $\langle \rho\alpha + \xi, \rho\beta + \xi \rangle$ or $\langle \rho\beta + \xi, \rho\alpha + \xi \rangle$ according as ρ is positive or not, most of discussions can be reduced to the cases of positive (i.e., non-negative definite) operators.

A finite family $\{A_1, \cdots, A_n\}$ of positive operators is said to be γ-*decomposable* in case there is a finite family of positive operators, admitting possible multiplicity, such that the total sum is γI and every A_j is a sum of a suitable subfamily. The definition can be also stated in this way: there is a positive operator-valued, finitely additive measure, with total measure γI, on a Boolean algebra, whose range contains all A's. A family of positive operators is said to be γ-decomposable in case every finite subfamily is γ-decomposable.

Given a 1-decomposable family $\mathfrak{A} = \{A_\lambda : \lambda \in \Lambda\}$, consider the free Boolean algebra with Λ as the set of generators. By the 1-decomposability, for any finite subset $\{\lambda_1, \cdots, \lambda_n\}$ of indices there is a normalized, positive operator valued, finitely additive measure on the subalgebra generated by $\lambda_1, \cdots, \lambda_n$, which assigns each A_{λ_j} to λ_j. Since the subalgebra is homomorphic image of the whole algebra (see [2, p. 141]), the measure can be extended over the latter. Now standard arguments based on the weak compactness of the set of positive contractions show that $\mathfrak{A}$ is contained in the range of a normalized, positive operator valued, finitely additive measure on the Boolean algebra. Then the famous theorem of Naimark ([1], [3]) guarantees that a 1-decomposable family admits a commutative family of dilations, consisting of projections, so that it is of $\langle 0, 1 \rangle$ type.

Conversely if a finite family $\{A_1, \cdots, A_n\}$ of positive operators admits a commutative family $\{T_1, \cdots, T_n\}$ of dilations, consisting of positive contractions on a superspace, then the family of products $P\cdot B_1\cdots B_n\cdot P$'s gives a 1-decomposition of $\{A_1, \cdots, A_n\}$, where P is the projection onto $\mathfrak{H}$, and $B_j = T_j$ or $= I - T_j$, because the positivity of a product $B_1\cdots B_n$ is a consequence of the commutativity among T's. Summing up, we obtain

Theorem 1. *A family $\mathfrak{A}$ of operators is of $\langle\alpha, \beta\rangle$ type if and only if $\mathfrak{A} - \alpha I$ is $(\beta-\alpha)$-decomposable.*

If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are γ_1- and γ_2-decomposable respectively, both $\mathfrak{A}_1 \cup \mathfrak{A}_2$ and $\mathfrak{A}_1 + \mathfrak{A}_2$ are $(\gamma_1+\gamma_2)$-decomposable. Thus with use of appropriate homotheties, we can prove

Corollary. *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are of $\langle\alpha_1, \beta_1\rangle$ and $\langle\alpha_2, \beta_2\rangle$ type respectively, both $\mathfrak{A}_1 \cup \mathfrak{A}_2$ and $\mathfrak{A}_1 + \mathfrak{A}_2$ are of $\langle\alpha, \beta\rangle$ type, where either $\alpha = \min(\alpha_1, \alpha_2)$ and $\beta = \beta_1 + \beta_2 - \alpha$ or $\beta = \max(\beta_1, \beta_2)$ and $\alpha = \alpha_1 + \alpha_2 - \beta$.*

In particular, if the sum of norms of all members of $\mathfrak{A}$ is bounded by γ, it is of $\langle -2\gamma, 2\gamma\rangle$ type, because the family of positive (and negative) parts of members is obviously γ-decomposable (cf. [3]).

Theorem 2. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be of $\langle\alpha_1, \beta_1\rangle$ and $\langle\alpha_2, \beta_2\rangle$ types respectively. If every member of $\mathfrak{A}_2$ commutes with all of $\mathfrak{A}_1 \cup \mathfrak{A}_2$, then $\mathfrak{A}_1 \cup \mathfrak{A}_2$ is of $\langle\alpha, \beta\rangle$ type with $\alpha = \min(\alpha_1, \alpha_2)$ and $\beta = \max(\beta_1, \beta_2)$.*

In fact, by Theorem 1, the commutativity assumption, and inductive observation we can assume that $\mathfrak{A}_1$ is a finite family, say $\{A_1, \cdots, A_n\}$, and $\mathfrak{A}_2$ consist of a single member, say B, and further that $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = 1$. Let $\{C_1, \cdots, C_m\}$ be a 1-decomposition of $\{A_1, \cdots, A_n\}$, then D's defined by

$$D_{2j} = B^{\frac{1}{2}}\cdot C_j \cdot B^{\frac{1}{2}} \text{ and } D_{2j-1} = (I-B)^{\frac{1}{2}}\cdot C_j\cdot (I-B)^{\frac{1}{2}}$$

give a 1-decomposition of $\{A_1, \cdots, A_n, B\}$, because

$$A_i = B^{\frac{1}{2}}\cdot A_i \cdot B^{\frac{1}{2}} + (I-B)^{\frac{1}{2}}\cdot A_i \cdot (I-B)^{\frac{1}{2}}$$

by the commutativity of A_i with $B^{\frac{1}{2}}$ and $(I-B)^{\frac{1}{2}}$.

A pair $\{A, B\}$ of positive contractions is of $\langle -1, 1\rangle$ type, because $\{I-A, I-B\}$ is of $\langle 0, 2\rangle$ type. $\{A, B\}$ is, however, not necessarily of $\langle 0, 1\rangle$ type. In this respect the following theorem is of some interest.

Theorem 3. *Let A and B be a projection and a positive contraction respectively. If $\{A, B\}$ is of $\langle 0, 1\rangle$ type, A commutes with B.*

In fact, since $\{A, B\}$ is 1-decomposable, there are positive operators C and D such that

$$0 \le C \le A, \quad 0 \le D \le I - A, \quad \text{and} \quad B = C + D.$$

Since $Af = 0$ implies $Cf = 0$ and A is a projection, it follows $C = C\cdot A$, hence A commutes with C. In a similar way, $I-A$ commutes with D. Thus A commutes with B.

References

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[3] B. Sz.-Nagy: Extensions of linear transformations in Hilbert space which extend beyond this space; Appendix to "Functional Analysis" by F. Riesz and B. Sz.-Nagy, New York (1960).