149. *On Indefinite (E.R.)-Integrals. I*

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§ 1. I.A. Vinogrdova [1] constructed a function $f(x)$ such that (i) $f(x)$ is $\mathcal{D}$-integrable [2] on [0, 1], (ii) $f(x)$ has a continuous indefinite A-integral, $A(x)=(A)\int_0^x f(t)dt$ [3], (iii) $A(x)\neq(\mathcal{D})\int_0^x f(t)dt$ ($x\in P$, mes $P>0$). On the other hand I. Amemiya and T. Ando [4] proved that A-integral is equivalent to (E.R.) integral for Lebesgue measure [5].

In the paper "On indefinite (E.R.)-integrals. II", we shall show that, for every function $f(x)$ which is $\mathcal{D}$-integrable on $I_0=[a, b]$, there exists a measure ν such that $f(x)$ has a indefinite (E.R. ν)-integral, (E.R. $\nu)\int_a^x f(t)dt$ [6], and (E.R. $\nu)\int_a^x f(t)dt=(\mathcal{D})\int_a^x f(t)dt$ for all $x\in I_0$.

For this purpose, first we shall generalize (see the Lemma of § 2) the theorem which has been proved by S. Nakanishi (formerly S. Enomoto) [7].

Nakanishi's theorem. *Let $f(x)$ be a function which is $\mathcal{D}^*$-integrable [8] on $I_0=[a, b]$ and let $F(I)=(\mathcal{D}^*)\int_I f(x)dx$. Then, for every sequence $\{\varepsilon_n\}$, $\varepsilon_n\downarrow 0$, there exists a non-decreasing sequence of closed sets such that (i) $\bigcup_{n=1}^{\infty} F_n=I_0$, (ii) $f(x)$ is summable on every F_n, (iii) the condition, $I_i\cap F_n\neq\phi$ for all i, implies that*

$$\left|\sum_{i=1}^{i_0} F(I_i)-\sum_{i=1}^{i_0}(L)\int_{I_i\cap F_n} f(x)dx\right|<\varepsilon_n$$

for every finite family $\{I_i : i=1\cdots i_0\}$ of non-overlapping intervals contained in I_0.

§ 2. For $\mathcal{D}$-integral, we shall prove the following lemma which may be regarded as a generalization of Nakanishi's theorem.

Lemma. *Let $f(x)$ be a function which is $\mathcal{D}$-integrable on $I_0=[a, b]$ and let $F(I)=(\mathcal{D})\int_a^x f(t)dt$. Then, for every sequence $\{\varepsilon_n\}$, $\varepsilon_n\downarrow 0$, there exists a non-decreasing sequence of closed sets $\{F_n\}$ such that (i) $\bigcup_{n=1}^{\infty} F_n=I_0$, (ii) $f(x)$ is summable on every F_n, (iii) $\left|F(I)-\int_{F_n\cap I} f(x)dx\right|\leq\varepsilon_n$ for every interval $I\subset I_0$, (iv) $\sum_{i=1}^{\infty}|F(I_n^i)|\leq\varepsilon_n$ for the sequence of intervals contiguous to the closed set which consists of all points of F_n and end points of I_0.*

Proof. It is enough to show that every function of $\mathcal{L}_\alpha(I_0)$, for $\alpha<\Omega$, possesses this property. Let us show it by transfinite induction.

(1) In the case, $\alpha=0$, being $\mathcal{L}_0(I_0)=\mathcal{L}(I_0)$, the sequence $\{F_n=I_0\}$ fulfills the required conditions (i), (ii), (iii), and (iv).

(2) In the case, $0<\alpha<\Omega$, we shall show that every function of $\mathcal{L}_\alpha(I_0)$ possesses this property if every functions of $\mathcal{L}_\xi(I_0)$, for $\xi<\alpha$, possesses. We shall consider two cases separately.

(a) The function $f(x)\in\left(\sum_{\xi<\alpha}\mathcal{L}_\xi(I_0)\right)^{\sigma}$. Let $\{p_l : l=0\cdots l_0\}$, $p_l<p_{l+1}$, be the set consisting of all $\left(\sum_{\xi<\alpha}\mathcal{L}_\xi\right)$-singular points in I_0 of $f(x)$ and end points of I_0. Then on account of continuity of $F(I)=(\mathcal{D})\int_I f(x)dt$, there exists $\delta_n>0$ such that, for every interval $I\subset I_0$,

$$|I|<\delta_n \qquad \text{implies} \qquad |F(I)|<\varepsilon_n/l_0 . \tag{1}$$

In each interval $[p_{l-1}, p_l]$, we may choose two sequences $\{a_l^j\}$ and $\{b_l^j\}$, $p_{l-1}<\cdots<a_l^{j+1}<a_l^j<\cdots<a_l^1<b_l^1<\cdots<b_l^j<b_l^{j+1}<\cdots<p_l$, such that

$$a_l^j-p_{l-1}<\delta_j \qquad \text{and} \qquad p_l-b_l^j<\delta_j \tag{2}$$

for every j.

We write $J_l^2=[a_l^1, b_l^1]$, $J_l^{2j-1}=[a_l^j, a_l^{j-1}]$ $(j=2, 3, \cdots)$,
$J_l^{2j}=[b_l^{j-1}, b_l^j]$ $(j=2, 3, \cdots)$,
$J_l'^{2j-1}=[p_{l-1}, a_l^j]$ $(j=1, 2, \cdots)$,
and $J_l'^{2j}=[b_l^j, p_l]$ $(j=1, 2, \cdots)$.

Since $f(x)\in\sum_{\xi<\alpha}\mathcal{L}_\xi(J_l^j)$ for every l and j, $f(x)\in\mathcal{L}_{\xi_l^j}(J_l^j)$ for some $\xi_l^j<\alpha$, and there exists, by hypothesis of induction, a non-decreasing sequence of closed sets $\{F_{l,m}^j$ $(m=1, 2, \cdots)\}$ such that (i′) $\bigcup_{m=1}^{\infty}F_{l,m}^j=I_0$, (ii′) $f(x)$ is summable on every $F_{l,m}^j$,

$$\text{(iii}'\text{)} \quad \left|F(I)-\int_{F_{l,m}^j} f(x)dx\right|\le\varepsilon^m/l_0\cdot 2^j \text{ for every interval } I\subset J_l^j, \tag{3}$$

$$\text{(iv}'\text{)} \quad \sum_{k=1}^{\infty}|F(I_{l,m}^{j,k})|\le\varepsilon_m/l_0\cdot 2^j \tag{4}$$

for the sequence of intervals $\{I_{l,m}^{j,k}$ $(k=1, 2, \cdots)\}$ contigouous to the closed set which consists of all points of $F_{l,m}^j$ and end points of J_l^j.

Writing $F_n=\bigcup_{l=1}^{l_0}\bigcup_{j=2}^{2n}F_{l,n}^j\cup\{p_l\}\cup\{a_l^j, b_l^j; (l=1, 2, \cdots l_0)(j=1, 2, \cdots n)\}$ we shall show that the sequence of closed sets F_n fulfills the required conditions (i), (ii), (iii), and (iv). It is clear by the construction of F_n that $\{F_n\}$ is a non-decreasing sequence of closed sets on each of which $f(x)$ is summable and that $\bigcup_{n=1}^{\infty}F_n=I_0$. On account of (1), (2), and (3), we have, for every interval $I\subset I_0$,

$$\left|F(I)-\int_{F_n\cap I}f(x)dx\right|=\left|\sum_{l=1}^{l_0}\left\{\sum_{j=2}^{2n}F(J_l^j\cap I)+F(J_l'^{2n-1}\cap I)+F(J_l'^{2n}\cap I)\right.\right.$$

$$-\sum_{j=2}^{2n}\int_{F^i_{l,n}\cap I} f(x)dx\Big\} \le \sum_{l=1}^{l_0}\sum_{j=2}^{2n}\left|F(J_l^j\cap I)-\int_{F^j_{l,n}\cap I} f(x)dx\right|$$

$$+\sum_{l=1}^{l_0}\{|F(J_l'^{2n-1}\cap I)|+|F(J_l'^{2n}\cap I)|\}$$

$$\le \sum_{l=1}^{l_0}\left(\sum_{j=2}^{2n}\varepsilon_n/l_0\cdot 2^j+\varepsilon_n/4l_0+\varepsilon_n/4l_0\right)<\varepsilon_n.$$

Let $\{I_n^i\}$ be the sequence of intervals contiguous to the closed set which consists of all points of F_n and end points of I_0. Then, the family $\{I_n^i\}$ is equal to the family

$$\{I_{l,n}^{j,k}(k=1,2,\cdots)\ (l=1,2,\cdots,l_0)\ (j=1,2,\cdots,2n)\}$$
$$\cup\{J_l'^{2n-1}(l=1,2,\cdots,l_0)\}\cup\{J_l'^{2n}(l=1,2,\cdots l_0)\}$$

and therefore, we have, by (1), (2), and (4),

$$\sum_{i=1}^{\infty}|F(I_n^i)|=\sum_{l=1}^{l_0}\left\{\sum_{j=2}^{2n}\sum_{k=1}^{\infty}|F(I_{l,n}^{j,k})|+|F(J_l'^{2n-1})|+|F(J_l'^{2n})|\right\}$$
$$\le \sum_{l=1}^{l_0}\left(\sum_{j=2}^{2n}\varepsilon_n/l_0\cdot 2^j+\varepsilon_n/4l_0+\varepsilon_n/4l_0\right)<\varepsilon_n.$$

(b) The function $f(x)\in(\sum_{\xi<\alpha}\mathcal{L}_\xi(I_0))^{\sigma H}$. Let S be the closed set of all $(\sum_{\xi<\alpha}\mathcal{L}_\xi)^{\sigma}$-singular points in I_0 of $f(x)$, $\{J_l(l=1,2,3\cdots)\}$ the sequence of intervals contiguous to the closed set consisting of all points of S and end points of I_0. Then $f(x)\in(\sum_{\xi<\alpha}\mathcal{L}_\xi(J_l))^{\sigma}$. Hence, by what has just been proved in (a), there exists a non-decreasing sequence of closed set $\{F_{l,m}(m=1,2,3\cdots)\}$ such that (i″) $\bigcup_{m=1}^{\infty}F_{l,m}=J_l$, (ii″) $f(x)$ is summable on $F_{l,m}$,

(iii″) $$\left|F(I)-\int_{F_{l,m}\cap I}f(x)dx\right|\le\varepsilon_m/2^{l+2} \tag{5}$$

for every interval $I\subset J_l$,

(iv″) $$\sum_{j=1}^{\infty}|F(I_{l,m}^j)|\le\varepsilon_m/2^{l+2} \tag{6}$$

for the sequence of intervals $\{I_{l,m}^j(j=1,2,\cdots)\}$ contiguous to the closed set which consists of all points of $F_{l,m}$ and end points of J_l.

Since $f(x)\in(\sum_{\xi<\alpha}\mathcal{L}_\xi(I_0))^{\sigma H}$, there exsists a strictly increasing sequence of integer $\{l_n\}$ such that

$$\sum_{l=l_n+1}^{\infty}|((\sum_{\xi<\alpha}\mathcal{L}_\xi)^{\sigma},\ J_l;\ f)|\le\varepsilon_n/8, \tag{7}$$

$$O((\sum_{\xi<\alpha}\mathcal{L}_\xi)^{\sigma},\ J_l;\ f)\le\varepsilon_n/8 \tag{8}$$

for all $l>l_n$.

Let $F_n'=\bigcup_{l=1}^{l_n}F_{l,n}$. Then $f(x)$ is summable on F_n'. Hence there exists $\delta_n'>0$ such that, for every measurable set $E\subset F_n'$,

$$\text{mes}(E)<\delta_n' \quad\text{implies}\quad \int_E|f(x)|dx<\varepsilon_n/8. \tag{9}$$

Let $f_s(x)$ be the restriction of f(x) on S and let

$$F_s(I)=(\mathscr{D})\int_I f_s(x)dx.$$

Then $f_s(x)\in(\sum_{\xi<\alpha}\mathscr{L}_\xi(I_0))^\sigma$ and there exists a non-decreasing sequence of closed sets $\{F_{0,m}\ (m=1,2\cdots)\}$ such that (i‴) $\bigcup_{m=1}^{\infty}F_{0,m}=I_0$, (ii‴) $f_s(x)$ is summable on $F_{0,m}$,

$$\text{(iii‴)}\qquad \left|F_s(I)-\int_{F_{0,m}\cap I}f(x)dx\right|\leq\varepsilon_m/4 \tag{10}$$

for every interval $I\subset I_0$,

$$\text{(iv‴)}\qquad \sum_{j=1}^{\infty}|F_s(I_{0,m}^j)|\leq\varepsilon_m/4 \tag{11}$$

for the sequence of intervals $\{I_{0,m}^j(j=1,2,\cdots)\}$ contiguous to the closed set which consists of all points of $F_{0,m}$ and end points of I_0.

Since $\lim_{m\to\infty}$ mes $(I_0-F_{0,m})=0$, we may assume that

$$\text{mes }(I_0-F_{0,m})<\delta'_m. \tag{12}$$

Writing $F_n=(S\cup F'_n)\cap F_{0,n}$ we shall show that the sequence of closed sets $\{F_n\}$ fulfills the required conditions (i), (ii), (iii), and (iv). It is clear that $\{F_n\}$ is non-decreasing sequence of closed sets on each of which $f(x)$ is summable, and that $\bigcup_{n=1}^{\infty}F_n=I_0$. Since $F_n=(S\cap F_{0,n})\cup[F'_n-\{F'_n\cap C(F_{0,n})\}]$ and mes $(S\cap F'_n)=0$, it follows from (5), (7), (8), (10), (9), and (12) that

$$\begin{aligned}\left|F(I)-\int_{F_n\cap I}f(x)dx\right| &= \left|\left\{F_s(I)+\sum_{l=1}^{\infty}F(J_l\cap I)\right\}\right.\\ &\quad\left.-\left\{\int_{S\cap F_{0,n}\cap I}f(x)dx+\int_{F'_n\cap I}f(x)dx-\int_{F'_n\cap C(F_{0,n})\cap I}f(x)dx\right.\right|\\ &\leq\left|F_s(I)-\int_{F_{0,n}\cap I}f(x)dx\right|+\sum_{l=1}^{l_n}\left|F(J_l\cap I)-\int_{F_{l,n}\cap I}f(x)dx\right|\\ &\quad+\sum_{l=l_n+1}^{\infty}|F(J_l\cap I)|+\int_{F'_n\cap C(F_{0,n})}|f(x)|dx\\ &\leq\varepsilon_n/4+\sum_{l=1}^{l_n}\varepsilon_n/2^{l+2}+\sum_{l=l_n+1}^{\infty}|F(J_l)|\\ &\quad+2\sup_{l>l_n}0((\sum_{\xi<\alpha}\mathscr{L}_\xi)^\sigma,\ J_l;\ f)+\varepsilon_n/8<\varepsilon_n\end{aligned}$$

for ever interval $I\subset I_0$.

Finally, let $\{I_n^j\}$ be the sequence of intervals contiguous to the closed set which consists of all points of F_n and end points of I_0. Then for each $I_{l,n}^j$, $0\leq l\leq l_n$, [for each $J_l(l>l_n)$], there exists a interval I_n^i such that $I_{l,n}^j\subset I_n^i[J_l\subset I_n^i]$. Hence, we have

$$I_n^i=(\bigcup_{\substack{1\leq l\leq l_n\\ I_{l,n}^j\subset I_n^i}}I_{l,n}^j)\cup(\bigcup_{\substack{l>l_n\\ J_l\subset I_n^i}}J_l)\cup\{\bigcup_{I_{0,n}^j\subset I_n^i}S\cap I_{0\,n}^j)\}$$
$$\cup(F'_n\cap C(F_{0,n})\cap I_n^i).$$

Therefore, on account of (6), (7), (11), (9), and (12), we have

$$\sum_{i=1}^{\infty}|F(I_n^i)|\leq\sum_{l=1}^{l_n}\sum_{j=1}^{\infty}|F(I_{l.n}^j)|+\sum_{l=l_n+1}^{\infty}|F(J_l)|$$

$$+\sum_{j=1}^{\infty}|F_s(I_{0,n}^j)|+\int_{F'_n\cap O(F_{0,n})}|f(x)|dx$$
$$\leq\varepsilon_n/4+\varepsilon_n/8+\varepsilon_n/4+\varepsilon_n/8<\varepsilon_n.$$

This complete the proof.

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