148. *Semigroups with a Maximal Homomorphic Image having Zero*[*)]

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Let S be a semigroup and suppose that S is homomorphic onto a semigroup S' with zero. Then S' is called a Z-homomorphic image of S. A Z-homomorphic image S_0 of S is called a maximal Z-homomorphic image of S if any Z-homomorphic image S' of S is a homomorphic image of S_0. An ideal T of a semigroup S is called a minimal ideal if it does not properly contain an ideal of S. Of course, a mininal ideal is unique if it exists. If S has a minimal ideal, S has a maximal Z-homomorphic image, but this converse is not true as Example 2 shows. This paper gives a necessary and sufficient condition for a semigroup to have a maximal Z-homomorphic image.

Let I be an ideal of a semigroup S. Then S/I denotes the Rees factor semigroup. The following lemmas are fundamental (cf. [1]).

Lemma 1. *Let I_1 and I_2 be ideals. If $I_1 \subseteq I_2$ then S/I_1 is homomorphic onto S/I_2.*

Lemma 2. *Let S' be any Z-homomorphic image of a semigroup S. Then there exists an ideal I of S such that S/I is homomorphic onto S'.*

Lemma 3. *If $I_2 \subset I_1$ and if S/I_1 is homomorphic onto S/I_2 then there is an ideal I_3 of S such that $I_1 \subset I_3$ and S/I_3 is homomorphic onto S/I_1.*

Let $\mathfrak{S}$ be the family of all ideals of a semigroup S. Hereafter, we shall call a subfamily of $\mathfrak{S}$ a family of ideals.

Theorem 1. *A semigroup S has a maximal Z-homomorphic image if and only if there is a non-empty family $\mathfrak{F}$ of ideals such that the following conditions are satisfied.*

(1.1) *If $I_\xi \in \mathfrak{F}$ and $I_\eta \in \mathfrak{S}$ such that $I_\eta \subseteq I_\xi$ then $I_\eta \in \mathfrak{F}$.*

(1.2) *If $I_\xi, I_\eta \in \mathfrak{F}$, and $I_\eta \subseteq I_\xi$, then S/I_ξ is homomorphic onto S/I_η.*

Proof. Necessity of (1.1) and (1.2): Suppose that S has a maximal Z-homomorphic image S_0. By Lemma 2, we may assume that $S_0 = S/I_0$ where I_0 is an ideal of S. $\mathfrak{F}$ is defined to be the system

*) The abstract of this paper was partly announced in [3] by one of the authors.

of all ideals I of S such that $I \subseteq I_0$. Clearly $\mathcal{F}$ satisfies (1.1). To show (1.2), take $I_\xi, I_\eta \in \mathcal{F}$ such that $I_\eta \subseteq I_\xi$. Since $I_\eta \subseteq I_\xi \subseteq I_0$, S/I_ξ is homomorphic onto S/I_0 by Lemma 1. On the other hand, since S/I_0 is a maximal Z-homomorphic image of S, S/I_0 is homomorphic onto S/I_η and hence S/I_ξ is homomorphic onto S/I_η. Therefore $\mathcal{F}$ satisfies (1.1) and (1.2).

Sufficiency of (1.1) and (1.2): Suppose that (1.1) and (1.2) are satisfied by $\mathcal{F}$. Let I_ξ be any element of $\mathcal{F}$. We shall prove that S/I_ξ is a maximal Z-homomorphic image of S. Let S' be any Z-homomorphic image of S. By Lemma 2, S/J is homomorphic onto S' for some ideal J of S. Let $I_\eta = J \cap I_\xi$. Clearly $\emptyset \neq I_\eta \subseteq I_\xi$. By (1.1), $I_\eta \in \mathcal{F}$. Since $I_\eta \subseteq I_\xi$, S/I_ξ is homomorphic onto S/I_η by (1.2), and S/I_η is homomorphic onto S/J because $I_\eta \subseteq J$. Therefore S/I_ξ is homomorphic onto S/J, hence onto S'. This completes the proof.

Thus we know that if a family $\mathcal{F}$ satisfies (1.1) and (1.2), then for every I_ξ of $\mathcal{F}$, S/I_ξ is a maximal Z-homomorphic image of S. Such a family $\mathcal{F}$ is called a normal family (of ideals) of S.

Suppose that a semigroup S has at least one normal family of ideals. Let $\mathfrak{N}$ denote the system of all non-empty normal families of ideals of S: $\mathfrak{N} = \{\mathcal{F}_\alpha : \alpha \in \Xi\}$. By the definition, we immediately have

(2.1) If $\mathcal{F}_\alpha \in \mathfrak{N}$, $\alpha \in \Lambda \subseteq \Xi$, then the union $\bigcup_{\alpha \in \Lambda} \mathcal{F}_\alpha \in \mathfrak{N}$

(2.2) If $\mathcal{F}_\alpha \in \mathfrak{N}$, $\alpha \in \Lambda \subseteq \Xi$, then the intersection $\bigcap_{\alpha \in \Lambda} \mathcal{F}_\alpha \in \mathfrak{N}$ if it is not empty.

Let $\mathcal{F}$ be a normal family of ideals and $\mathcal{G}$ be a subfamily of $\mathcal{F}$ such that

$$I_\xi \in \mathcal{G},\ I_\eta \in \mathcal{F}, \quad \text{and} \quad I_\eta \subseteq I_\xi \quad \text{imply} \quad I_\eta \in \mathcal{G}$$

$\mathcal{G}$ is called a lower ideal of $\mathcal{F}$. Then we have

(2.3) If $\mathcal{F} \in \mathfrak{N}$, any lower ideal of $\mathcal{F}$ is also in $\mathfrak{N}$. By a principal family generated by I_{ξ_0} in $\mathcal{F}$ we mean a family of all ideals $I_\xi \in \mathfrak{S}$ such that $I_\xi \subseteq I_{\xi_0}$ where I_{ξ_0} is a fixed element of $\mathcal{F}$. Clearly any principal family in $\mathcal{F}$ is a lower ideal of $\mathcal{F}$ and hence a normal family by (1.1) and (1.2).

$\mathfrak{N}$ contains a unique maximal element $\mathcal{F}_1$, $\mathcal{F}_1 = \bigcup_{\alpha \in \Xi} \mathcal{F}_\alpha$, the union of all normal families; $\mathcal{F}_1$ is the set of all ideals I of S such that S/I is a maximal Z-homomorphic image of S.

Theorem 2. *Let S be a semigroup having a maximal Z-homomorphic image, and let $\mathfrak{N} = \{\mathcal{F}_\alpha; \alpha \in \Xi\}$ be the system of all non-empty normal families. Then the following statements are equivalent.*

(3.1) *S has a minimal ideal.*

(3.2) *$\bigcap_{\alpha \in \Xi} \mathcal{F}_\alpha$ consists of exactly one ideal.*

(3.3) $\bigcap_{\alpha \in \Xi} \mathcal{F}_\alpha$ *is not empty.*

(3.4) *There is a normal family* $\mathcal{F}$ *such that* $\mathcal{F}$ *consists of exactly one ideal.*

Proof. (3.1)→(3.2), (3.1)→(3.4): If I_0 is a minimal ideal of S, then $\mathcal{F}=\{I_0\}$ is a normal family. Since $I_0 \subseteq I$ for all ideals I, $\mathcal{F}$ is contained in any normal family: $\bigcap_{\alpha \in \Xi} \mathcal{F}_\alpha=\{I_0\}$.

(3.2)→(3.3): Trivial. Now we shall prove (3.3)→(3.1). Suppose that S has no minimal ideal. Let $J_1 \in \mathcal{F}_0=\bigcap_{\alpha \in \Xi} \mathcal{F}_\alpha$. Since S has no minimal ideal, there is an ideal J_2 of S such that J_2 is properly contained in J_1. For $J_i (i=1, 2)$ let $\mathcal{G}_i$ denote the principal family generated by $J_i (i=1, 2)$. Each $\mathcal{G}_i$ is a normal family, and J_1 is in $\mathcal{G}_1$ but not in $\mathcal{G}_2$;

$$\mathcal{G}_2 \subset \mathcal{G}_1 \subseteq \mathcal{F}_0.$$

This contradicts the fact that $\mathcal{F}_0$ is a minimal normal family.

(3.4)→(3.1): Suppose that a normal family $\mathcal{F}$ consists of I_0 alone. If I_0 contains properly an ideal I of S, then $\mathcal{F}$ contains I besides I_0 by (1.1). This is a contradiction. Hence we have (3.1). Thus the theorem has been proved.

Corollary. *Let S be a semigroup with a maximal Z-homomorphic image. If S has no minimal ideal, there exists an infinite properly ascending chain of ideals of S*

(4) $\cdots \supset I_n \supset \cdots \supset I_2 \supset I_1$

such that S/I_n is a maximal Z-homomorphic image for each positive integer n.

Proof. By theorem 1 there is a normal family $\mathcal{F}$ of ideals. Let I_2 be one of the elements of $\mathcal{F}$. If S has no minimal ideal, there is an I_1 such that $I_1 \subset I_2$. Since $I_2 \in \mathcal{F}$, we see $I_1 \in \mathcal{F}$ and S/I_2 is homomorphic onto S/I_1. By Lemma 3, there is an ideal I_3 such that $I_3 \supset I_2$ and S/I_3 is homomorphic onto S/I_2. By repeated process, we have an infinite properly ascending chain of ideals.

The converse of the corollary is not true as Example 1 shows. We shall give a few examples without detailed proof.

Example 1. This is an example of a semigroup that has a minimal ideal and yet has an infinite properly ascending chain of ideals satisfying the condition of Corollary.

Let S be the set of symbols:

$$S=\{(i, j): i=0, 1, 2, \cdots; \text{ if } i=0, \text{ then } j=0, 1; \text{ if } i > 0, \text{ then } j=1,2,3,4\}$$

and let $I_i=\{(k, j); k \leqq i\}$ and $\bar{I}_i=I_i \setminus \bigcup_{k<i} I_k$.

We define an operation in S as follows:

$$(0, 0)^2=(0, 1)^2=(0, 0), (0, 0)(0, 1)=(0, 1)(0, 0)=(0, 1)$$

if $i\neq 0$, then $(i,j)(0,l)=(0,l)(i,j)=(0,l)$, $l=0,1$; if $i\neq 0, k\neq 0, i\neq k$, then $(i,j)(k,l)=(0,0)$, $l=0,1$. The product $(i,j)(i,l)$ in $\bar{I}_i$, $i>1$, is given by

	$(i,1)$	$(i,2)$	$(i,3)$	$(i,4)$
$(i,1)$	$(i,1)$	$(i,2)$	$(0,0)$	$(0,0)$
$(i,2)$	$(0,0)$	$(0,0)$	$(i,1)$	$(i,2)$
$(i,3)$	$(i,3)$	$(i,4)$	$(0,0)$	$(0,0)$
$(i,4)$	$(0,0)$	$(0,0)$	$(i,3)$	$(i,4)$

(I_i/I_{i-1} $(i=1,2,\cdots)$ is a semigroup without proper homomorphism.)

Then it is easily seen that S is a semigroup and all I_i's are ideals of S

$$I_0\subset I_1\subset I_2\subset\cdots\subset I_i\subset\cdots$$

I_0 is a group and a minimal ideal of S and $S/I_0\cong S/I_i$ $(i=1,2,\cdots)$ which is a maximal Z-homomorphic image of S.

Example 2. This is an example of a semigroup that has a maximal Z-homomorphic image and yet has no minimal ideal.

Let S be the set of symbols

$$\{\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots\}$$

together with an operation defined by the rule that if a_i and a_j are in S, then $a_i a_j = a_k$ where k is the minimum of the integers i and j. Then S is a semigroup under this operation and the following properties hold for S.

1. Each proper ideal I_n of S has the form
$$I_n=\{\cdots, a_{n-2}, a_{n-1}, a_n\}\,,$$
$$n=\cdots, -2, -1, 0, 1, 2, \cdots$$
2. For any two integers m and n the semigroups S/I_m and S/I_n are isomorphic.
3. For each integer n, S/I_n is a maximal Z-homomorphic image of S.

Example 3. This is an example of a semigroup with at least two non-isomorphic maximal Z-homomorphic images.

Let S be the set of symbols

$$\{a_0, a_1, a_2, \cdots\}$$

together with an operation defined by the rule that if a_i and a_j are in S, then $a_i a_j = a_k$ where k is the largest non-negative even integer less than or equal to the minimum of the integers i and j. Then S is a semigroup with a zero a_0 and the following properties hold for S.

1. For each non-negative integer n, the set $I_n=\{a_0, a_1, \cdots, a_n\}$ is an ideal of S and there exists a homomorphism of S/I_n onto S. Thus S/I_n is a maximal Z-homomorphic image of S.
2. If n is a non-negative integer then S/I_n is isomorphic onto S if and only if n is even.

3. If m and n are non-negative integers then S/I_m and S/I_n are isomorphic if and only if m and n are both even or both odd.

Addendum: After writing this paper, we have found that Theorems 1 and 2 can be extended to a general case, maximal homomorphic images of a given type, with a slight modification. The detailed discussion will be published elsewhere.

References

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