180. Note on Permutability of Congruences on Algebraic Systems

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In his paper [3, Theorem 4], A. I. Mal'cev gave a necessary and sufficient condition that all congruences should be permutable for every algebraic system of a primitive class: There must exist a derived composition $f(\xi, \eta, \zeta)$ (a function defined by iteration of the compositions) such that $f(\xi, \xi, \zeta)=\zeta$ and $f(\xi, \zeta, \zeta)=\xi$ are identities in each of this class. This condition is clearly equivalent to the following: Let $\mathfrak{F}$ be the algebraic system freely generated by $\{x, y, z\}$ in this class. If φ_1 and φ_2 are congruences on $\mathfrak{F}$ generated by the relations $x\equiv y$ and $y\equiv z$ respectively, then there exists a derived composition $f(\xi, \eta, \zeta)$ such that $f(x, y, z)$ and z are congruent modulo φ_1 and that $f(x, y, z)$ and x are congruent modulo φ_2. In his book [1, pp. 22–23], R. H. Bruck has stated that the Mal'cev's result does not apply to multiplicative quasigroups, and that the free quasigroup of rank 4 (and hence any free quasigroup of higher rank) has non-permutable multiplicative congruences, but the facts for free quasigroups of rank 1, 2, or 3 seem to be unknown, similarly for free loops of arbitrary positive rank.

In this note, we shall study generalizations of the above Mal'cev's result and the others. Theorem 1 is a generalization of the Mal'cev's result, which can apply to multiplicative quasigroups. By this theorem, we can easily obtain that the free quasigroup of rank 3 has non-permutable multiplicative congruences. Theorems 2 and 3 are similar generalizations of the analogous results [2, Theorems 1 and 2] for weak permutability and local permutability of congruences. These theorems can apply to multiplicative loops, and it can be easily seen that all multiplicative congruences on any loop are locally permutable.

Let $\mathfrak{A}$ be an algebraic system with respect to a system V of compositions, and let W be a subsystem of V. An equivalence relation θ on $\mathfrak{A}$ is called a W-congruence if and only if

$$w(a_1, a_2, \cdots, a_{N(w)}) \overset{\theta}{\sim} w(b_1, b_2, \cdots, b_{N(w)})^{1)}$$

holds for every composition w in W, and for all elements $a_1, a_2, \cdots,$

1) $x \overset{\theta}{\sim} y$ denotes that x and y are congruent modulo θ.

$a_{N(w)}$, and $b_1, b_2, \cdots, b_{N(w)}$ in $\mathfrak{A}$ such that

$$a_1 \overset{\theta}{\sim} b_1,\ a_2 \overset{\theta}{\sim} b_2,\ \cdots,\ a_{N(w)} \overset{\theta}{\sim} b_{N(w)},$$

where w is an $N(w)$-ary composition.

Now let θ_1 and θ_2 be two W-congruences on $\mathfrak{A}$. We shall consider the following conditions between θ_1 and θ_2 at an element a in $\mathfrak{A}$:

$K_1(a)$ For each element x in $\mathfrak{A}$, there exists an element y in $\mathfrak{A}$ satisfying $a \overset{\theta_1}{\sim} y \overset{\theta_2}{\sim} x$ if and only if there exists an element z in $\mathfrak{A}$ satisfying $a \overset{\theta_2}{\sim} z \overset{\theta_1}{\sim} x$.

$K_2(a)$ If $x \overset{\theta_1}{\sim} a \overset{\theta_2}{\sim} y$, then there exists an element z in $\mathfrak{A}$ satisfying $x \overset{\theta_2}{\sim} z \overset{\theta_1}{\sim} y$.

θ_1 and θ_2 are said to be weakly permutable at a if and only if θ_1 and θ_2 satisfy the condition $K_1(a)$. θ_1 and θ_2 are said to be locally permutable at a if and only if θ_1 and θ_2 satisfy both $K_1(a)$ and $K_2(a)$. θ_1 and θ_2 are said to be permutable if and only if θ_1 and θ_2 satisfy $K_2(a)$ for every element a in $\mathfrak{A}$.

It is easily verified that θ_1 and θ_2 are permutable if and only if θ_1 and θ_2 are weakly permutable at every element a in $\mathfrak{A}$, and if and only if θ_1 and θ_2 are locally permutable at every element a in $\mathfrak{A}$.

Lemma 1. *Let A_V be a set of composition-identities with respect to a system V of compositions, and let W be a subsystem of V. Let $\mathfrak{F}$ be a free A_V-algebraic system $F(\{x_1, x_2, \cdots, x_n\}, A_V)$,*[2] *and let R be a set of relations $p_i(x_1, x_2, \cdots, x_n) \equiv q_i(x_1, x_2, \cdots, x_n)$,*[3] *$i \in I$. And let φ be the W-congruence on $\mathfrak{F}$ generated by R, i.e. the least W-congruence on $\mathfrak{F}$ satisfying $p_i(x_1, x_2, \cdots, x_n) \overset{\varphi}{\sim} q_i(x_1, x_2, \cdots, x_n)$ for all $i \in I$. Moreover let $\mathfrak{A}$ be an A_V-algebraic system generated by a set $\{a_1, a_2, \cdots, a_n\}$ of n generators, and let θ be a W-congruence on $\mathfrak{A}$ satisfying $p_i(a_1, a_2, \cdots, a_n) \overset{\theta}{\sim} q_i(a_1, a_2, \cdots, a_n)$ for all $i \in I$. Then, for every pair of derived compositions $f(\xi_1, \xi_2, \cdots, \xi_n)$ and $g(\xi_1, \xi_2, \cdots, \xi_n)$ with respect to V,*

$$f(x_1, x_2, \cdots, x_n) \overset{\varphi}{\sim} g(x_1, x_2, \cdots, x_n)$$

$$\textit{implies } f(a_1, a_2, \cdots, a_n) \overset{\theta}{\sim} g(a_1, a_2, \cdots, a_n)\ .$$

Proof. Let θ' be the equivalence relation on $\mathfrak{F}$ such that

$$P(x_1, x_2, \cdots, x_n) \overset{\theta'}{\sim} Q(x_1, x_2, \cdots, x_n)$$

if and only if $P(a_1, a_2, \cdots, a_n) \overset{\theta}{\sim} Q(a_1, a_2, \cdots, a_n)$. Then θ' is clearly a W-congruence on $\mathfrak{F}$, and satisfies $p_i(x_1, x_2, \cdots, x_n) \overset{\theta'}{\sim} q_i(x_1, x_2, \cdots, x_n)$

2) $F(\{x_1, x_2, \cdots, x_n\}, A_V)$ denotes the free A_V-algebraic system generated by the free set $\{x_1, x_2, \cdots, x_n\}$ of generators.

3) Elements in a free A_V-algebraic system $F(\{x_1, x_2, \cdots, x_n\}, A_V)$ are denoted by $f(x_1, x_2, \cdots, x_n)$, $p(x_1, x_2, \cdots, x_n)$, $P(x_1, x_2, \cdots, x_n)$ etc., by using derived compositions with respect to V.

for all $i \in I$. That is, θ' is a W-congruence on $\mathfrak{F}$ satisfying R. Hence $\theta' \geqq \varphi$, because φ is the least W-congruence on $\mathfrak{F}$ satisfying R. Now suppose that $f(x_1, x_2, \cdots, x_n) \overset{\varphi}{\sim} g(x_1, x_2, \cdots, x_n)$. Then $f(x_1, x_2, \cdots, x_n) \overset{\theta'}{\sim} g(x_1, x_2, \cdots, x_n)$, because $\theta' \geqq \varphi$. Hence $f(a_1, a_2, \cdots, a_n) \overset{\theta}{\sim} g(a_1, a_2, \cdots, a_n)$ follows from the definition of θ'. This completes the proof.

Theorem 1. *Let A_V be a set of composition-identities with respect to a system V of compositions, and let W be a subsystem of V. Then the following two propositions are equivalent:*

(α) *Any two W-congruences on any A_V-algebraic system are permutable.*

(β) *There exists an element $f(x, y, z)$ in the free A_V-algebraic system $F(\{x, y, z\}, A_V)$ such that*

$$f(x, y, z) \overset{\varphi_1}{\sim} z \quad \text{and} \quad f(x, y, z) \overset{\varphi_2}{\sim} x\ ,$$

where φ_1 and φ_2 are the W-congruences on $F(\{x, y, z\}, A_V)$ generated by the relations $x \equiv y$ and $y \equiv z$ respectively.

Before we state the proof of this theorem, we shall note the

Remark. It can be easily seen that the proposition (β) is equivalent to the following proposition:

(β') Let ψ_1 and ψ_2 be W-congruences on $F(\{x, y, z\}, A_V)$. If $x \overset{\psi_1}{\sim} y \overset{\psi_2}{\sim} z$, then there exists an element y' in $F(\{x, y, z\}, A_V)$ such that $x \overset{\psi_2}{\sim} y' \overset{\psi_1}{\sim} z$.

Hence by Theorem 1, the permutability of W-congruences for all A_V-algebraic systems can be reduced to that of W-congruences on the free A_V-algebraic system $F(\{x, y, z\}, A_V)$.

Proof. The implication (α)$\Rightarrow$(β) immediately follows from the first part of the above remark. Hence we shall prove the converse implication (β)$\Rightarrow$(α). Let $\mathfrak{A}$ be any A_V-algebraic system, and let θ_1 and θ_2 be any two W-congruences on $\mathfrak{A}$. Let a, b, and c be elements in $\mathfrak{A}$ such that $a \overset{\theta_1}{\sim} b \overset{\theta_2}{\sim} c$. It is sufficient to prove that there exists an element d in $\mathfrak{A}$ satisfying $a \overset{\theta_2}{\sim} d \overset{\theta_1}{\sim} c$. Now let $\mathfrak{B}$ be the V-subsystem of $\mathfrak{A}$ generated by the elements a, b, and c, i.e. the least subset of $\mathfrak{A}$ containing a, b, and c which is closed with respect to V. Then it is easy to see that $\mathfrak{B}$ is an A_V-algebraic system, and that $\theta_1(\mathfrak{B})$[4] and $\theta_2(\mathfrak{B})$ are W-congruences on $\mathfrak{B}$ and satisfy $a \overset{\theta_1(\mathfrak{B})}{\sim} b$ and $b \overset{\theta_2(\mathfrak{B})}{\sim} c$ respectively. Now let φ_1 and φ_2 be the W-congruences on $F(\{x, y, z\}, A_V)$ generated by the relations $x \equiv y$ and

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4) $\theta_1(\mathfrak{B})$ denotes the equivalence relation on $\mathfrak{B}$ such that $b_1 \overset{\theta_1(\mathfrak{B})}{\sim} b_2$ if and only if $b_1 \overset{\theta_1}{\sim} b_2$.

$y\equiv z$ respectively. Then by (β), there exists an element $f(x, y, z)$ in $F(\{x, y, z\}, A_V)$ such that $x \overset{\varphi_2}{\sim} f(x, y, z) \overset{\varphi_1}{\sim} z$. Hence by Lemma 1, $a \overset{\theta_2(\mathfrak{B})}{\sim} f(a, b, c) \overset{\theta_1(\mathfrak{B})}{\sim} c$. That is, there exists an element d $(=f(a, b, c))$ in $\mathfrak{A}$ satisfying $a \overset{\theta_2}{\sim} d \overset{\theta_1}{\sim} c$. This completes the proof.

Example 1. By the above theorem, we know the fact that the free quasigroup of rank 3 has non-permutable multiplicative congruences. For if any two multiplicative congruences on the free quasigroup of rank 3 are permutable, then by Theorem 1 or Remark, any two multiplicative congruences on any quasigroup must be permutable. But H. A. Thurston [4] showed the fact that the free commutative quasigroup of rank 4 has non-permutable multiplicative congruences.

Lemma 2. *Let V be a system of compositions which contains a 0-ary composition e, and let A_V be a set of composition-identities with respect to V. Let W be a subsystem of V. Then the following two propositions are equivalent:*

(1) *Any two W-congruences θ_1 and θ_2 on any A_V-algebraic system $\mathfrak{A}$ satisfy the condition $K_1(e)$.*[5]

(2) *Let φ_1 and φ_2 be two W-congruences on the free A_V-algebraic system $F(\{x, y\}, A_V)$. If φ_1 and φ_2 are generated by the relations $y\equiv e$ and $x\equiv y$ respectively, then there exists an element $g(x, y)$ in $F(\{x, y\}, A_V)$ such that*

$$g(x, y) \overset{\varphi_1}{\sim} x \quad \textit{and} \quad g(x, y) \overset{\varphi_2}{\sim} e\,.$$

Proof. The implication (1)$\Rightarrow$(2) is obvious. Hence we shall prove the converse implication (2)$\Rightarrow$(1). Suppose that a and b are elements in $\mathfrak{A}$ satisfying $e \overset{\theta_1}{\sim} b \overset{\theta_2}{\sim} a$. Now let $\mathfrak{B}$ be the V-subsystem of $\mathfrak{A}$ generated by the elements a and b. Then it is easy to see that $\mathfrak{B}$ is an A_V-algebraic system, and that $\theta_1(\mathfrak{B})$ and $\theta_2(\mathfrak{B})$ are W-congruences on $\mathfrak{B}$ and satisfy $b \overset{\theta_1(\mathfrak{B})}{\sim} e$ and $a \overset{\theta_2(\mathfrak{B})}{\sim} b$ respectively. Let φ_1 and φ_2 be the W-congruences on $F(\{x, y\}, A_V)$ which are generated by the relations $y\equiv e$ and $x\equiv y$ respectively. Then by (2), there exists an element $g(x, y)$ in $F(\{x, y\}, A_V)$ satisfying $e \overset{\varphi_2}{\sim} g(x, y) \overset{\varphi_1}{\sim} x$. Hence by Lemma 1, $e \overset{\theta_2(\mathfrak{B})}{\sim} g(a, b) \overset{\theta_1(\mathfrak{B})}{\sim} a$. That is, there exists an element c $(=g(a, b))$ in $\mathfrak{A}$ satisfying $e \overset{\theta_2}{\sim} c \overset{\theta_1}{\sim} a$. Conversely if there exists an element c in $\mathfrak{A}$ satisfying $e \overset{\theta_2}{\sim} c \overset{\theta_1}{\sim} a$, then we can similarly obtain that there exists an element b in $\mathfrak{A}$ satisfying $e \overset{\theta_1}{\sim} b \overset{\theta_2}{\sim} a$. This completes the proof.

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5) The 0-ary composition e can be considered as a constant element in every A_V-algebraic system.

Lemma 3. *Under the same conditions as in Lemma 2, the following two propositions are equivalent:*

(3) *Any two W-congruences θ_1 and θ_2 on any A_V-algebraic system $\mathfrak{A}$ satisfy the condition $K_2(e)$.*

(4) *Let φ_1 and φ_2 be W-congruences on the free A_V-algebraic system $F(\{x, y\}, A_V)$. If φ_1 and φ_2 are generated by the relations $y \equiv e$ and $x \equiv e$ respectively, then there exists an element $h(x, y)$ in $F(\{x, y\}, A_V)$ such that*

$$h(x, y) \overset{\varphi_1}{\sim} x \quad \text{and} \quad h(x, y) \overset{\varphi_2}{\sim} y.$$

Proof. The implication (3)⇒(4) is obvious. Hence we shall prove the converse implication (4)⇒(3). Suppose that a and b are elements in $\mathfrak{A}$ satisfying $a \overset{\theta_2}{\sim} e \overset{\theta_1}{\sim} b$. Now let $\mathfrak{B}$ be the V-subsystem of $\mathfrak{A}$ generated by a and b. Then it is easy to see that $\mathfrak{B}$ is an A_V-algebraic system, and that $\theta_1(\mathfrak{B})$ and $\theta_2(\mathfrak{B})$ are W-congruences on $\mathfrak{B}$ and satisfy $b \overset{\theta_1(\mathfrak{B})}{\sim} e$ and $a \overset{\theta_2(\mathfrak{B})}{\sim} e$ respectively. Let φ_1 and φ_2 be the W-congruences on $F(\{x, y\}, A_V)$ generated by the relations $y \equiv e$ and $x \equiv e$ respectively. Then by (4), there exists an element $h(x, y)$ in $F(\{x, y\}, A_V)$ satisfying $x \overset{\varphi_1}{\sim} h(x, y) \overset{\varphi_2}{\sim} y$. Hence by Lemma 1, $a \overset{\theta_1(\mathfrak{B})}{\sim} h(a, b) \overset{\theta_2(\mathfrak{B})}{\sim} b$. That is, there exists an element c $(=h(a, b))$ in $\mathfrak{A}$ satisfying $a \overset{\theta_1}{\sim} c \overset{\theta_2}{\sim} b$. This completes the proof.

Changing the expression of Lemma 2, we have the

Theorem 2. *Let V be a system of compositions which contains a 0-ary composition e, and let A_V be a set of composition-identities with respect to V. Let W be a subsystem of V. Then the following two propositions are equivalent:*

(α^*) *Any two W-congruences on any A_V-algebraic system are weakly permutable at e.*

(β^*) *There exists an element $g(x, y)$ in the free A_V-algebraic system $F(\{x, y\}, A_V)$ such that*

$$g(x, y) \overset{\varphi_1}{\sim} x \quad \text{and} \quad g(x, y) \overset{\varphi_2}{\sim} e,$$

where φ_1 and φ_2 are the W-congruences on $F(\{x, y\}, A_V)$ generated by the relations $y \equiv e$ and $x \equiv y$ respectively.

Combining Lemmas 2 and 3, we have the

Theorem 3. *Under the same conditions as in Theorem 2, the following two propositions are equivalent:*

($\alpha^\#$) *Any two W-congruences on any A_V-algebraic system are locally permutable at e.*

($\beta^\#$) *There exist two elements $g(x, y)$ and $h(x, y)$ in the free A_V-algebraic system $F(\{x, y\}, A_V)$ such that*

$$g(x, y) \overset{\varphi_1}{\sim} x,\ g(x, y) \overset{\varphi_2}{\sim} e \quad \text{and} \quad h(x, y) \overset{\varphi_1}{\sim} x,\ h(x, y) \overset{\varphi_3}{\sim} y,$$

where φ_1, φ_2, and φ_3 are the W-congruences on $F(\{x, y\}, A_V)$ generated by the relations $y\equiv e$, $x\equiv y$, and $x\equiv e$ respectively.

Example 2. Let V be the system of binary compositions $\cdot$, $/$, $\backslash$ and of a 0-ary composition e. And let A_V be the set of composition-identities with respect to V which consists of $(\xi/\eta)\cdot\eta=\xi$, $\xi\cdot(\xi\backslash\eta)=\eta$, $(\xi\cdot\eta)/\eta=\xi$, $\xi\backslash(\xi\cdot\eta)=\eta$, and of $\xi\cdot e=\xi$, $e\cdot\xi=\xi$. Then it is clear that A_V defines loops.

Now let W be the subsystem of V which consists of only the composition $\cdot$. And let φ_1 and φ_2 be the W-congruences, i.e. the multiplicative congruences on the free loop $F(\{x, y\}, A_V)$ which are generated by the relations $y\equiv e$ and $x\equiv y$ respectively. Then there exists an element $g(x, y)$ in $F(\{x, y\}, A_V)$ such that

$$g(x, y)\overset{\varphi_1}{\sim}x \quad \text{and} \quad g(x, y)\overset{\varphi_2}{\sim}e,$$

because $x\cdot(y\backslash e)=x\cdot(e\cdot(y\backslash e))\overset{\varphi_1}{\sim}x\cdot(y\cdot(y\backslash e))=x\cdot e=x$ and $x\cdot(y\backslash e)\overset{\varphi_2}{\sim}y\cdot(y\backslash e)=e$. Hence by Theorem 2, any two multiplicative congruences on any loop are weakly permutable at e.

Moreover let φ_3 be the multiplicative congruence on $F(\{x, y\}, A_V)$ generated by the relation $x\equiv e$. Then there exists an element $h(x, y)$ in $F(\{x, y\}, A_V)$ such that

$$h(x, y)\overset{\varphi_1}{\sim}x \quad \text{and} \quad h(x, y)\overset{\varphi_3}{\sim}y,$$

because $x\cdot y\overset{\varphi_1}{\sim}x\cdot e=x$ and $x\cdot y\overset{\varphi_3}{\sim}e\cdot y=y$. Hence by Theorem 3, any two multiplicative congruences on any loop are locally permutable at e.

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