179. *On Branching Markov Processes*

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In this paper we give a definition of branching Markov processes in terms of a property of the semi-group of Markov processes and describe several equivalent formulations. Our definition is a generalization of Kolmogoroff and Dmitriev's one [2] and would clarify the situation discussed in Skorohod [4] (cf. [3]). This equivalence plays an important role in studies of branching Markov processes.

1. Definition of branching Markov processes. Let S be a compact Hausdorff space satisfying the axiom of the second countability, therefore it is metrizable. We denote the metric of S by ρ_1. Let us denote by $S^{(n)}$ the n-fold product of S with product topology and the symmetrization of $S^{(n)}$ by S^n, i.e. S^n is the quotient space $S^{(n)}/R$ of $S^{(n)}$ by the equivalence relation R of permutation with quotient topology, therefore S^n is also metrizable. We denote the metric of S^n by ρ_n. Moreover, $S^0=\{\partial\}$, where ∂ is an extra point. This procedure is due to Moyal [3].

Let γ be the natural mapping from $\bigcup_{n=0}^{\infty} S^{(n)}$ to $\bigcup_{n=0}^{\infty} S^n$. We introduce a metric ρ in $\bigcup_{n=0}^{\infty} S^n$ defined by the formula

$$\rho(x, y)=\begin{cases}\dfrac{1}{2}\dfrac{\rho_n(x, y)}{1+\rho_n(x, y)}, & \text{if } x, y \in S^n,\\ |n-m|, & \text{if } x \in S^n,\ y \in S^m, \text{ and } n \neq m.\end{cases} \tag{1.1}$$

We denote the one-point compactification of $\bigcup_{n=0}^{\infty} S^n$ by $\boldsymbol{S}=\bigcup_{n=0}^{\infty} S^n \cup \{\Delta\}$.

In the following, $\boldsymbol{B}(S)$ (resp. $\boldsymbol{B}(\boldsymbol{S})$) is the space of bounded and Borel measurable functions on S (resp. $\boldsymbol{S}$) and $\boldsymbol{B}^*(S)$ is the subset of $\boldsymbol{B}(S)$ formed by functions f with $\|f\|\leqq 1$. We define a mapping $\wedge$ from $f\in \boldsymbol{B}^*(S)$ to $\hat{f}\in \boldsymbol{B}(\boldsymbol{S})$ by

$$\hat{f}(\boldsymbol{x})=\begin{cases}1, & \text{if } \boldsymbol{x}=\partial,\\ f(x_1)f(x_2)\cdots f(x_n), & \text{if } \boldsymbol{x}\in S^n \text{ and } \boldsymbol{x}\ni(x_1, x_2, \cdots, x_n),\\ 0, & \text{if } \boldsymbol{x}=\Delta.\end{cases} \tag{1.2}$$

Definition 1.1. A strong Markov process $X=\{x_t(w), \zeta, N_t, P_{\boldsymbol{x}}\}$[1)] on $\boldsymbol{S}$ is said to be a *branching Markov process* if its semi-group $\{T_t; t\geqq 0\}$ satisfies

1) Cf. e.g. [1]. We always assume that paths are right continuous and have left limits, and $\zeta=\infty$.

(1.3) $$T_t\hat{f}=\widehat{(T_t\hat{f}|_S)}, \text{ for any } f\in \boldsymbol{B}^*(S).$$[2),3)]

We now introduce several notations:

$\xi_t(w)=n$, if $x_t(w)\in S^n$.

$e_\Delta(w)=\inf\{t; x_t(w)=\Delta\}$, $(\inf \phi=\infty)$.

$e_\partial(w)=\inf\{t; x_t(w)=\partial\}$, $(\inf \phi=\infty)$.

$\tau(w)=\inf\{t; \xi_0(w)\neq\xi_t(w)\}$, $(\inf \phi=\infty)$.

$\tau_0(w)=0$, $\tau_1(w)=\tau(w)$, and $\tau_n(w)=\tau_{n-1}(w)+\theta_{\tau_{n-1}}\tau(w)$ $(n\geqq 2)$.

Then ξ_t, e_Δ, e_∂, and τ are able to be interpreted as, and will be called, the number of particles, explosion time, the extinction time and the first branching time of the process, respectively.

2. Fundamental properties of branching Markov processes. We shall study detailed structures of the branching Markov processes defined in the previous section. In this section $X=\{x_t, \zeta, N_t, P_x\}$ is a strong Markov process on S and not assumed apriori to be a branching Markov process.

Lemma 2.1. If X is a branching Markov process, then it satisfies

Condition (c.1): For any $\boldsymbol{x}\in \boldsymbol{S}$,

(i) $P_{\boldsymbol{x}}[x_t=\partial$ for any $t\geqq s$ if $x_s=\partial]=1$, and

(ii) $P_{\boldsymbol{x}}[x_t=\Delta$ for any $t\geqq s$ if $x_s=\Delta]=1$.

And if X has the quasi-left continuity, it satisfies

(c.2) $$P_{\boldsymbol{x}}\Big[\lim_{n\to\infty}\tau_n=e_\Delta, \lim_{n\to\infty}\tau_n<\infty\Big]=P_{\boldsymbol{x}}\Big[\lim_{n\to\infty}\tau_n<\infty\Big].$$

We now prepare some notations. First we shall extend the natural mapping γ as follows:

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_m\in \boldsymbol{S}$. Then there happens just one of the following three cases:

1) $\boldsymbol{x}_i=\Delta$ for some i,

2) $\boldsymbol{x}_i=\partial$ for every i,

3) All $\boldsymbol{x}_i$ are different from Δ and there is some $\boldsymbol{x}_i$ different from ∂.

We shall define

(2.1) $$\gamma(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_m)=\begin{cases}\Delta, \text{ if } 1^\circ) \text{ holds},\\ \partial, \text{ if } 2^\circ) \text{ holds},\\ \gamma(x_{11}, x_{12}, \cdots, x_{1n_1}, \cdots, x_{m1}, x_{m2}, \cdots, x_{mn_m}),\\ \quad \text{if } 3^\circ) \text{ holds},\end{cases}$$

where we take $(x_{11}, x_{12}, \cdots, x_{1n_1})\in \boldsymbol{x}_1, \cdots, (x_{m1}, x_{m2}, \cdots, x_{mn_m})\in \boldsymbol{x}_m$ except such j as $\boldsymbol{x}_j=\partial$. Then γ defines a mapping from $\bigcup_{n=1}^{\infty}\boldsymbol{S}^{(n)}$ to $\boldsymbol{S}$, where $\boldsymbol{S}^{(n)}$ is the n-fold product of $\boldsymbol{S}$.

Remark. In the simplest case where S consists of a single point we can identify S^n with n and Δ with $+\infty$, and the above mapping

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2) For $f\in \boldsymbol{B}(\boldsymbol{S})$, $f|_S$ denotes the restriction of f on S.

3) We need to modify the present definition slightly to include wider class of branching processes in our discussion.

γ is given by

$$\gamma(n_1, n_2, \cdots, n_m)=n_1+n_2+\cdots+n_m.$$

Let W be the fundamental space of X on which P_x is defined. We are now going to define a mapping ϕ by which $\bar{w}\in \bar{W}=\bigcup_{n=1}^{\infty} W^{(n)}$[4] corresponds to a path $\tilde{x}_t(\bar{w})=(\phi\bar{w})(t)$ on S: for $(w^1, \cdots, w^n)=\bar{w}\in \bar{W}$,

(2.2) $\tilde{x}_t(\bar{w})=(\phi\bar{w})(t)=\gamma[x_t(w^1), x_t(w^2), \cdots, x_t(w^n)]$, for all $t\geqq 0$.

Let $\tilde{N}_t$ (resp. $\tilde{N}$) be the σ-field on $\bar{W}$ generated by $\tilde{x}_s(\bar{w})$ $(s\leqq t)$ (resp. $s\geqq 0$) and define a system of probability measures $\tilde{P}_{\boldsymbol{x}}$, $\boldsymbol{x}\in \boldsymbol{S}$ on $(\bar{W}, \tilde{N})$ by putting: for $\boldsymbol{x}\in S^n$, $(x_1, x_2, \cdots, x_n)\in \boldsymbol{x}$

$$\text{(2.3)}\qquad \tilde{P}_{\boldsymbol{x}}(A)=\begin{cases}P_{x_1}\times P_{x_2}\times\cdots\times P_{x_n}(A), & \text{if } A\in\tilde{N} \text{ and } A\subset W^{(n)},\\ 0, & \text{if } A\in\tilde{N} \text{ and } A\subset W^{(m)}\ (n\neq m),\end{cases}$$

and for $\boldsymbol{x}=\partial$ or Δ,

$$\tilde{P}_{\boldsymbol{x}}(A)=\begin{cases}P_{\boldsymbol{x}}(A), & \text{if } A\in\tilde{N} \text{ and } A\subset W,\\ 0, & \text{if } A\in\tilde{N} \text{ and } A\subset W^{(m)},\ (m\neq 1).\end{cases}$$

Proposition 2.1. The value $\tilde{P}_{\boldsymbol{x}}[A]$ is independent of representatives $(x_1, x_2, \cdots, x_n)\in \boldsymbol{x}$ for $A\in \tilde{N}$ and so it is well defined.

Definition 2.1. A process $X=\{x_t, \zeta, N_t, P_{\boldsymbol{x}}\}$ is said to have property B. I, if it has the following

PROPERTY B. I. The processes $\{\tilde{x}_t, \tilde{\zeta}, \tilde{N}_t, \tilde{P}_{\boldsymbol{x}}\}$ and $\{x_t, \zeta, N_t, P_{\boldsymbol{x}}\}$ are equivalent.

We need additional preparations before stating property B. II. Let us introduce the following notations;

$$\text{(2.4)}\qquad \tau_n^*(w)=\begin{cases}\tau(w), & \text{if } x_0(w)\in S^n,\\ 0, & \text{if } x_0(w)\notin S^n.\end{cases}$$

$$\text{(2.5)}\qquad x_t^*(w)=\begin{cases}x_t(w), & \text{if } t<\tau_n^*(w),\\ x_{\tau_n^*}(w), & \text{if } t\geqq\tau_n^*(w).\end{cases}$$

Then we denote the stopped process[5] of X at τ_n^* by

(2.6) $X_n=\{x_t^*, \tau_n^*, N_t, P_{\boldsymbol{x}}\}$, $(\boldsymbol{x}\in \boldsymbol{S})$.

Let us suppose that

(2.7) $Y^{(k)}=\{y_t^{(k)}, \tau^{(k)}, N_t^{(k)}, P_x^{(k)}\}$ $(k=1, 2, \cdots, n)$

are equivalent processes with X_1. We denote the fundamental space of $Y^{(k)}$ by Ω_k and its element by $w^{(k)}$, and introduce the following quantities;

(2.8) $\bar{\Omega}^{(n)}=\Omega_1\times\Omega_2\times\cdots\times\Omega_n$,

(2.9) $\bar{y}_t(\bar{w})=(y_t^{(1)}(w^{(1)}), y_t^{(2)}(w^{(2)}), \cdots, y_t^{(n)}(w^{(n)}))$, for $\bar{w}=(w^{(1)}, w^{(2)}, \cdots, w^{(n)})\in\Omega^{(n)}$.

(2.10) $\tau^*(\bar{w})=\min\{\tau^{(1)}(w^{(1)}), \tau^{(2)}(w^{(2)}), \cdots, \tau^{(n)}(w^{(n)})\}$.

$$\text{(2.11)}\qquad \bar{y}_t^*(\bar{w})=\begin{cases}\bar{y}_t(\bar{w}), & \text{if } t<\tau^*(\bar{w}),\\ \bar{y}_{\tau^*}(\bar{w}), & \text{if } t\geqq\tau^*(\bar{w}).\end{cases}$$

(2.12) $y_t^*(\bar{w})=\gamma\bar{y}_t^*(\bar{w})$.

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4) $W^{(n)}=W\times\cdots\times W$.

5) Cf. e.g. [1]. Where $x_{\tau_n^*}^*$ is the stopped point.

$$(2.13) \qquad Y_n^* = \{y_t^*, \tau^*, N_t^{(1)} \times \cdots \times N_t^{(n)}, \bar{P}_{\boldsymbol{x}} = P_{x_1}^{(1)} \times \cdots \times P_{x_n}^{(n)}\}, \quad \text{for } (x_1, x_2, \cdots, x_n) \in \boldsymbol{x}, \boldsymbol{x} \in S^n.$$

We extend Y_n^* for $\boldsymbol{x} \notin S^n$ so as

$$\bar{P}_{\boldsymbol{x}}[y_t^*(\bar{w}) = y_0^*(\bar{w}), \text{ for any } t \geqq 0] = 1.$$

Proposition 2.2. The process Y_n^* given by (2.13) is well-defined as a process on $\boldsymbol{S}$.

Definition 2.2. A process X is said to have property B. II, if it has the following

PROPERTY B. II. The processes X_n and Y_n^* defined in (2.6) and (2.13) are equivalent for any $n = 1, 2, \cdots$.

Definition 2.3. A process X is said to have property B. III, if it has the following

PROPERTY B. III. For any $f \in \boldsymbol{B}^*(S)$ and $\boldsymbol{x} \in S^n$,

(i) $E_{\boldsymbol{x}}[\hat{f}(x_t); t < \tau] = \prod_{j=1}^{n} E_{x_j}[\hat{f}(x_t); t < \tau]$, and

(ii) $E_{\boldsymbol{x}}[\hat{f}(x_\tau); \tau \in dt] = \sum_{k=1}^{n} E_{x_k}[\hat{f}(x_\tau); \tau \in dt] \prod_{j \neq k} E_{x_j}[f(x_t); t < \tau]$,

hold, where $(x_1, x_2, \cdots, x_n) \in \boldsymbol{x}$ and $n = 2, 3, \cdots$.

We now state fundamental

Theorem 1. *Let X be a strong Markov process on $\boldsymbol{S}$ subject to a condition*

(c.3) $\qquad P_{\boldsymbol{x}}[\tau(w) = s] = 0$, *for any* $s \geqq 0$ *and* $\boldsymbol{x} \in \boldsymbol{S}$.

Consider the following statements:

(i) *X is a branching Markov process;*

(ii) *X has the property B. I and satisfies* (c.1)

(iii) *X has the property B. II and satisfies* (c.1)

(iv) *X has the property B. III and satisfies* (c.1).

Then (i) *and* (ii) *are mutually equivalent,* (ii) *implies* (iii) *and* (iii) *implies* (iv), *and if X satisfies* (c.2) *then* (iv) *implies* (i).

3. Lemmas. Proofs that (ii)$\Rightarrow$(iii)$\Rightarrow$(iv) are rather straightforward. The proof that (i)$\Rightarrow$(ii) needs the following

Lemma 3.1. Let X be a branching Markov process. Take $\boldsymbol{x} \in S^n$ $(n = 1, 2, \cdots)$; $m_1, m_2, \cdots, m_q = 0, 1, 2, \cdots,$; $0 \leqq t_1 < t_2 < \cdots < t_q$; and $f_1, f_2, \cdots, f_q \in \boldsymbol{B}^*(S)$ $(q = 1, 2, 3, \cdots)$. Then

$$(3.1) \qquad E_{\boldsymbol{x}}[\hat{f}_1(x_{t_1})\hat{f}_2(x_{t_2}) \cdots \hat{f}_q(x_{t_q}); \xi_{t_1} = m_1, \xi_{t_2} = m_2, \cdots, \xi_{t_q} = m_q]$$
$$= \sum_{(1)} \cdots \sum_{(q)} \prod_{k=1}^{n} E_{x_k}[\hat{f}_1(x_{t_1})\hat{f}_2(x_{t_2}) \cdots \hat{f}_q(x_{t_q}); \xi_{t_1} = p_k^{(1)}, \cdots, \xi_{t_q} = p_k^{(q)}]$$

where $(x_1, x_2, \cdots, x_n) \in \boldsymbol{x}$ and $\sum_{(j)}$ denotes the sum over all the choices of $(p_1^{(j)}, p_2^{(j)}, \cdots, p_n^{(j)})$ subject to $p_1^{(j)} + p_2^{(j)} + \cdots + p_n^{(j)} = m_j$ and $p_1^{(j)}, p_2^{(j)}, \cdots, p_n^{(j)} \geqq 0$.

The assertion that (iv) implies (i) is verified by several steps of lemmas as follows.

Lemma 3.2. Let X have the property B. III. Then

$$(3.2)\quad E_{\boldsymbol{x}}[\hat{f}(x_\tau);\ \tau\in dt,\ \xi_\tau=m] =\sum_{k=1}^{n} E_{x_k}[\hat{f}(x_\tau);\ \tau\in dt,\ \xi_\tau=m-(n-1)]\prod_{j\neq k} E_{x_j}[f(x_t);\ t<\tau],$$

where $f\in \boldsymbol{B}^*(S)$, $m\geqq n-1$, $m\neq n$, and $(x_1, x_2, \cdots, x_n)\in \boldsymbol{x}$, $\boldsymbol{x}\in S^n$.

Lemma 3.3. Let X satisfy the property B. III. Let $u_k(t, x)$ $(k=1, 2, \cdots, m)$ be bounded and measurable functions on $[0, \infty)\times S$. Then, for $(x_1, x_2, \cdots, x_n)\in \boldsymbol{x}$, $\boldsymbol{x}\in S^n$

$$E_{\boldsymbol{x}}\Big[\sum_{\Pi}\prod_{j=1}^{m} u_{\Pi(j)}(\tau, x_\tau^{(j)});\ \xi_\tau=m,\ \tau\in dt\Big]$$

$$(3.3)\quad =\sum_{k=1}^{n}\ \sum_{(q_1,\cdots,q_{m-n+1})}^{(m)}\ \sum_{\hat{\Pi}}^{(\hat{q})} E_{x_k}\Big[\sum_{\Pi}^{(q)}\prod_{j=1}^{m-n+1} u_{\Pi(q_j)}(\tau, x_\tau^{(j)});\ \xi_\tau=\mathrm{m}-\mathrm{n}+1,\ \tau\in dt\Big]\times\prod_{j\neq k} E_{x_j}[u_{\hat{\Pi}(\hat{q}_j)}(t, x_t);\ t<\tau],$$

where $(x_\tau^{(1)}(w), \cdots, x_\tau^{(m)}(w))\in x_\tau(w)$, $\sum_{(q_1,\cdots,q_{m-n+1})}^{(m)}$ denotes the sum over all the choices $(q_1, \cdots, q_{m-n+1})$ from $(1, 2, \cdots, m)$, $\sum_{\Pi}^{(q)}$ denotes the sum over all the permutations of $(q_1, \cdots, q_{m-n+1})$, and $\sum_{\hat{\Pi}}^{(\hat{q})}$ denotes the sum over all the permutations of $(\hat{q}_1, \cdots, \hat{q}_{n-1})$ which is the remainder of $(1, 2, \cdots, m)$ excluding $(q_1, \cdots, q_{m-n+1})$.

Lemma 3.4. Take $f\in \boldsymbol{B}^*(S)$. Then

$$(3.4)\quad \sum_{p_1+\cdots+p_n=p}\int_0^t \sum_{k=1}^{n} E_{x_k}[\hat{f}(x_t);\ \tau_{p_k+1}\leqq t<\tau_{p_k+2},\ \tau\in ds]\times\prod_{j\neq k} E_{x_j}[\hat{f}(x_t);\ \tau_{p_j}\leqq t<\tau_{p_j+1},\ s<\tau] =\sum_{\substack{p_1+\cdots+p_n=p+1\\ p_1,\cdots,p_n\geqq 0}}\prod_{j=1}^{n} E_{x_j}[\hat{f}(x_t);\ \tau_{p_j}\leqq t<\tau_{p_j+1}].$$

Lemma 3.5. Let X satisfy the property B. III. Let $f\in \boldsymbol{B}^*(S)$ and $(x_1, x_2, \cdots, x_n)\in \boldsymbol{x}$, $\boldsymbol{x}\in S^n$. Then for any $p\geqq 0$

$$(3.5)\quad E_{\boldsymbol{x}}[\hat{f}(x_t);\ \tau_p\leqq t<\tau_{p+1}]=\sum_{\substack{p_1+\cdots+p_n=p\\ p_1,\cdots,p_n\geqq 0}}\prod_{j=1}^{n} E_{x_j}[\hat{f}(x_t);\ \tau_{p_j}\leqq t<\tau_{p_j+1}].$$

Note that we have for $f\in \boldsymbol{B}^*(S)$, if X satisfies (c.1) and (c.2),

$$(3.6)\quad T_t\hat{f}(\boldsymbol{x})=\sum_{p=0}^{\infty} E_{\boldsymbol{x}}[\hat{f}(x_t);\ \tau_p\leqq t<\tau_{p+1}],$$

$$(3.7)\quad \prod_{j=1}^{n} T_t\hat{f}(x_j)=\sum_{p_1}\cdots\sum_{p_n}\prod_{j=1}^{n} E_{x_j}[\hat{f}(x_t);\ \tau_{p_j}\leqq t<\tau_{p_j+1}] =\sum_{p=0}^{\infty}\sum_{\substack{p_1+\cdots+p_n=p\\ p_1,\cdots,p_n\geqq 0}}\prod_{j=1}^{n} E_{x_j}[\hat{f}(x_t);\ \tau_{p_j}\leqq t<\tau_{p_j+1}].$$

Then by virtue of above lemmas we have, if X has property B. III, (c.1) and (c.2),

$$T_t\hat{f}(\boldsymbol{x})=\prod_{j=1}^{n} T_t\hat{f}(x_j),\ (x_1, \cdots, x_n)\in \boldsymbol{x},\ \boldsymbol{x}\in S^n,$$

i.e. X is a branching Markov process.

In the proofs of above lemmas the next combinatorial lemma is useful.

Lemma 3.6. Let $\{a_{ij}\}$ be a double sequence. Then

$$\sum_{\Pi}\prod_{j=1}^{m} a_{\Pi(j)j}=\prod_{j=1}^{m}\left(\sum_{k=1}^{m} a_{kj}\right)-\sum_{(k_1,\cdots,k_{m-1})}\prod_{j=1}^{m}\left(\sum_{q=1}^{m-1} a_{k_q j}\right)+\sum_{(k_1,\cdots,k_{m-2})}\prod_{j=1}^{m}\left(\sum_{q=1}^{m-2} a_{k_q j}\right)-\cdots(-1)^{m-1}\sum_{k}\prod_{j=1}^{m} a_{kj}, \tag{3.8}$$

where $\{\Pi(j)\}$ denotes a permutation of $\{j\}$, $\sum_{\Pi}$ denotes the sum over all permutations and $\sum_{(k_1,\cdots,k_r)}$ $(r\leqq m-1)$ denote the sums over all the choices for $(k_1, k_2, \cdots, k_r)$ from $(1, 2, \cdots, m)$.[6]

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6) Cf. e.g. [5].