177. Axiom Systems of B-algebra

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In my previous note [1], I gave an algebraic formulations of the classical propositional calculus, and I defined four algebraic systems called B, NB, BN, and NBN-algebras. In this note, we shall give other characterizations of B-algebra.

Let $\langle X, 0, *, \sim\rangle$ be an abstract algebra containing 0 as an element of a set X, where $*$ is a binary operation and $\sim$ is an unary operation on X. If $x*y=0$, then we shall denote it by $x\leqslant y$.

The axiom system of B-algebra is given by

$L\,1$ $x*y\leqslant x$,

$L\,2$ $(x*z)*(y*z)\leqslant(x*y)*z$,

$L\,3$ $x*y\leqslant(\sim y)*(\sim x)$,

$L\,4$ $0\leqslant x$.

If $x\leqslant y$ and $y\leqslant x$, then we define $x=y$. This axiom system is equivalent to axioms 1, 2, and

3′ $x=\sim(\sim x)$,

4′ $(\sim y)*(\sim x)\leqslant x*y$,

as already shown in [1]. Next we shall consider the following axiom system:

$H\,1$ $x*y\leqslant x$,

$H\,2$ $(x*y)*z\leqslant(x*z)*y$,

$H\,3$ $(x*y)*(x*z)\leqslant z*y$,

$H\,4$ $x*(\sim y)\leqslant y$,

$H\,5$ $x*(x*(\sim y))\leqslant x*y$,

$H\,6$ $0\leqslant x$.

We first prove some lemmas from axioms $H\,1\sim H\,6$.

By $H\,2$ and the definition of equality, we have

(1) $$(x*y)*z=(x*z)*y.$$

In $H\,2$, put $x=y, y=y*x, z=y*(\sim x)$, then by $H\,5$, we have

(2) $$x*(x*y)\leqslant x*(\sim y).$$

In (1), put $y=x$, $z=0$, then by $H\,1$, we have

$$(x*x)*0\leqslant(x*0)*x=0,$$

hence $x*x=0$.

(3) $$x*x=0, \text{ i.e. } x\leqslant x.$$

Let $x*z=z*y=0$, then by $H\,3$, we have $x*y=0$. This shows the following

(4) $$x\leqslant y,\ y\leqslant z \text{ imply } x\leqslant z.$$

By H 5 and (3), we have

(5) $$x*(x*(\sim x))=0, \text{ i.e. } x\leqslant x*(\sim x).$$

By (2) and (3), we have

(6) $$(\sim x)*((\sim x)*x)=0, \text{ i.e. } \sim x\leqslant(\sim x)*x.$$

Applying H 1, from (5) and (6), we have

(7) $$x=x*(\sim x), \qquad x=(\sim x)*x.$$

In (2), put $x=x*(\sim(\sim x))$, then

$$(x*(\sim(\sim x)))*((x*(\sim(\sim x)))*x)\leqslant(x*(\sim(\sim x)))*(\sim x).$$

The right side is equal to 0 by H 4, and further the second term of the left side is 0 by H 1. Hence we have $x*(\sim(\sim x))=0$. Therefore

(8) $$x\leqslant\sim(\sim x).$$

Let us put $x=\sim(\sim x)$, $y=\sim(\sim x)$, $z=x*(\sim x)$ in H 3, then we have

$$(\sim(\sim x)*x)*(\sim(\sim x)*(\sim(\sim x)*(\sim x)))\leqslant(\sim(\sim x)*(\sim x))*x.$$

Here $(\sim(\sim x)*(\sim x))*x=0$ by H 4, and $(\sim(\sim x)*(\sim(\sim x)*(\sim x)))=0$ by (7), then we have

(9) $$\sim(\sim x)\leqslant x.$$

By (8) and (9), we have

(10) $$\sim(\sim x)=x.$$

By H 3, if $z\leqslant y$, i.e. $z*y=0$, then $x*y\leqslant x*z$, hence

(11) $x\leqslant y$ implies $z*y\leqslant z*x$.

By (3) and the commutative law (1), we have $(x*y)*(z*y)\leqslant x*z$. Suppose $x*z=0$, then $x*y\leqslant z*y$. Hence

(12) $x\leqslant y$ implies $x*z\leqslant y*z$.

In H 2, if we put $y=\sim y$ and use (10), then we have $x*(x*y)\leqslant x*(\sim y)$. Next we substitute $x*y$ for x, and use $x*y\leqslant\sim y$ proved by H 4 and (10), then we have $(x*y)*((x*y)*y)=0$, and $x*y\leqslant (x*y)*y\leqslant x*y$ by H 1. Hence

(13) $$(x*y)*y=x*y.$$

Axiom 3 and the commutative law (1) imply $(x*z)*(y*z)\leqslant x*y$. If we operate z from the right of formula above, by (12), we have

$$((x*z)*(y*z))*z\leqslant(x*y)*z.$$

Applying the commutative law (1), we have

$$((x*z)*z)*(y*z)\leqslant(x*y)*z.$$

Axiom H 3 and (12) imply

$$((x*y)*(x*z))*u\leqslant(z*y)*u.$$

Using the commutative law into the left side, then we have

$$((x*y)*u)*(x*z)\leqslant(z*y)*u.$$

Now we substitute $x*z$ for x, $y*z$ for y, $(x*z)*z$ for z, and $(x*y)*z$ for u, then we have

$$(((x*z)*(y*z))*((x*y)*z))*((x*z)*((x*z)*z))$$
$$\leqslant(((x*z)*z)*(y*z))*((x*y)*z).$$

By the formula above, the right side is equal to 0, and by (13), the second term of the left side is 0, hence we have the following important

(14) $$(x*z)*(y*z)\leqslant(x*y)*z.$$

We substitute $u*x$ for x in (14), then

$$((u*x)*z)*(y*z)\leqslant((u*x)*y)*z.$$

By (2), we have

$$((u*z)*x)*(y*z)\leqslant((u*y)*x)*z\leqslant(u*y)*x.$$

Hence

(15) $$((u*z)*x)*(y*z)\leqslant(u*y)*x.$$

By axiom 3 and (11), we have

$$u*(y*x)\leqslant u*((z*x)*(z*y)).$$

Next we substitute $(z*(z*y))*(z*(z*x))$ for u, then

$$((z*(z*y))*(z*(z*x)))*(y*x)$$
$$\leqslant((z*(z*y))*(z*(z*x)))*((z*x)*(z*y)).$$

The right side is equal to 0 by $H\,2$, hence $(z*(z*y))*(z*(z*x))\leqslant y*x$. By the commutative law,

$$(z*(z*y))*(y*x)\leqslant z*(z*x).$$

Put $z=\sim x, y=x$ by $\sim x*((\sim x)*x)=0$. we have

(16) $$((\sim x)*((\sim x)*y))*(y*x)=0.$$

In (15), put $u=\sim x$, $z=\sim y$, $x=y*x$, $y=\sim x*y$, then

$$(((\sim x)*(\sim y))*(y*x))*(((\sim x)*y)*(\sim y))$$
$$\leqslant((\sim x)*((\sim x)*y))*(y*x).$$

By (16), the right side is 0, and $(\sim x)*y\leqslant\sim y$, we have

(17) $$(\sim x)*(\sim y)=y*x.$$

As shown in my note [1], (10) and (17) imply

(18) $$x*y=(\sim y)*(\sim x).$$

Therefore our axiom system defines a B-algebra.

The ideas of some deductions are stated in the case of the proofs of (H)$\Rightarrow$(F), (R), and (L_3) (see [2]).

In my note [1], any B-algebra (axioms $L\,1\sim L\,4$) is NB, BN, NBN-algebras and we proved that $\sim(\sim x)=x$ holds in the B-algebra.

Let $\langle X, 0, *, \sim\rangle$ be a B-algebra satisfying $L\,1\sim L\,4$. Then we shall prove that $H\,1\sim H\,6$ hold in the B-algebra.

If we substitute $(z*x)*(y*x)$ for x, $(z*y)*x$ for y and $z*y$ for z in axiom $L\,2$, then we have

$$(((z*x)*(y*x))*(z*y))*(((z*y)*x)*(z*y))$$
$$\leqslant(((z*x)*(y*x))*((z*y)*x))*(z*y)$$
$$=0*(z*y)=0.$$

by $L\,2$ and $L\,4$. Further, by $L\,1$ we have $((z*y)*x)*(z*y)=0$. Therefore,

$$(z*x)*(y*x)\leqslant z*y. \qquad \text{(a)}$$

By the proposition 4 in [1], we have

$$(z*x)*(z*y)\leqslant y*x,$$

which shows $H\,3$. Next we shall prove $H\,2$ from $L\,1$, $L\,2$, and $L\,4$. $y*x\leqslant y$ implies

$$(z*x)*y\leqslant(z*x)*(y*x)$$

(see the proposition 5 in [1]). In the formula (a), put $z=(x*y)*z$, $x=(x*z)*y$, and $y=(x*y)*(z*y)$, then

$$(((x*y)*z)*((x*z)*y))*(((x*y)*(z*y))*((x*z)*y))$$
$$\leqslant((x*y)*z)*((x*y)*(z*y)).$$

The right side is 0 by the formula above, and the second term of the left side is 0 by $L\,2$, hence we have $H\,2$:

$$(x*y)*z=(x*z)*y.$$

We proved that $L\,1\sim L\,4$ imply $H\,4$ (see proposition 8 in [1]). Now we shall prove that $H\,5$ is deduced from $L\,1\sim L\,4$. Since the algebra is *BN*, *NB*, *NBN*-algebras, it is sufficient to verify

$$\sim(x*y)*(\sim x)\leqslant y*(\sim x).$$

By $L\,2$, we have

$$\begin{aligned}&(\sim(x*y)*(\sim x))*(y*(\sim x))\\&\leqslant(\sim(x*y)*y)*(\sim x)\\&=((\sim y)*(x*y))*(\sim x)\\&=((\sim y)*(\sim x))*(x*y)\\&=(x*y)*(x*y)=0.\end{aligned}$$

Therefore we complete the proof of $H\,4$. Hence we have the following

Theorem. Any B-algebra is characterized by the axiom system $H\,1\sim H\,6$.

References

[1] K. Iséki: Algebraic formulation of propositional calculi. Proc. Japan Acad., **41**, 803-807 (1965).

[2] K. Iséki and S. Tanaka: On axiom systems of propositional calculi. X. Proc. Japan Acad., **41**, 801-802 (1965).