

174. On Axiom Systems of Propositional Calculi. IX

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In this note, we shall prove that the Frege axioms 1~5 of propositional calculus imply (H), (R), (L_1), (L_2), (L_3), (S_1), (S_2), and (M) axiom systems (for these notations and rules, see [2]). Y. Arai and K. Iséki, in their note [1], published that the (F)-system is equivalent to (L_3). They have shown that the (L_3)-system implies $CCNppp$. So, we do not prove that the (F)-system implies $CCNppp$, but we use this thesis.

- 1 $CpCqp.$
- 2 $CCpCqrCCpqCpr.$
- 3 $CCpqCNqNp.$
- 4 $CNNpp.$
- 5 $CpNNp.$
 - 1 $p/CCpCqrCCpqCpr, q/Cqr *C2-6,$
- 6 $CCqrCCpCqrCCpqCpr.$
 - 2 $p/Cqr, q/CpCqr, r/CCpqCpr *C6-C1 p/Cqr, q/p-7,$
- 7 $CCqrCCpqCpr.$
 - 2 $p/Cqr, q/Cpq, r/Cpr *C7-8,$
- 8 $CCCqrCpqCCqrCpr.$
 - 1 $p/CCCqrCpqCCqrCpr, q/Cpq *C8-9,$
- 9 $CCpqCCCqrCpqCCqrCpr.$
 - 2 $p/Cpq, q/CCqrCpq, r/CCqrCpr *C9-C1 p/Cpq,$
 $q/Cqr-10,$
- 10 $CCpqCCqrCpr.$
 - 10 $q/NNp, r/q *C5-11,$
- 11 $CCNNpqCpq.$
 - 7 $r/NNq *C5 p/q-12,$
- 12 $CCpqCpNNq.$
 - 7 $q/NNq *C4-13,$
- 13 $CCpNNqCpq.$
 - 10 $p/CNqp, q/CNpNNq, r/CNpq *C3 p/Nq, q/p-C13$
 $p/Np-14,$
- 14 $CCNpqCNqp.$
 - 10 $p/CNpNq, q/CNNqp, r/Cqp *C14 q/Nq-C11 p/q,$
 $q/p-15,$
- 15 $CCNpNqCqp.$
 - 1 $p/CCNqNpCpq, q/Np *C15 p/q, q/p-16,$
- 16 $CNqCCNqNpCpq.$

- 2 $p/Np, q/CNqNp, r/Cpq$ *C16—C1 $p/Np, q/Nq$ —17,
 17 $CNpCpq.$
 1 $p/CqCpq, q/CCpqCpr$ *C1 $p/q, q/p$ —18,
 18 $CCCpqCprCqCpq.$
 2 $p/CCpqCpr, q/CqCpq, r/CqCpr$ *C7 $p/q, q/Cpq,$
 r/Cpr —C18—19,
 19 $CCCpqCprCqCpr.$
 7 $p/CpCqr, q/CCpqCpr, r/CqCpr$ *C19—C2—20,
 20 $CCpCqrCqCpr.$
 20 $p/Np, q/p, r/q$ *C17—21,
 21 $CpCNpq.$
 10 $p/q, q/Cpq$ *C1 $p/q, q/p$ —22,
 22 $CCCpqrCqr.$
 1 $p/CpCrp$ *C1 q/r —23,
 23 $CqCpCrp.$
 20 $p/q, q/p, r/Crp$ *C23—24,
 24 $CpCqCrp.$
 10 $p/Cpq, q/CCqrCpr, r/s$ *C10—25,
 25 $CCCCqrCprCsCCpqs.$
 25 $q/Cqr, r/Csr, s/CCsqCpCsr$ *C25 $p/s, s/CpCsr$ —26,
 26 $CCpCqrCCsqCpCsr.$
 26 $p/Nq, s/p$ *C17 $p/q, q/r$ —27,
 27 $CCpqCNqCpr.$
 10 $p/CpNq, q/CNNqNp, r/CqNp$ *C3 q/Nq —C11 $p/q,$
 q/Np —28,
 28 $CCpNqCqNp.$
 10 $p/Np, q/Cpq$ *C17—29,
 29 $CCCpqrCNpqr.$
 20 $q/CrCsr, r/p$ *C1 $q/CrCsr$ —C1 $p/r, q/s$ —30,
 30 $Cpp.$
 25 $s/CCCprCsCCqrs$ *C10 $p/Cqr, q/Cpr, r/s$ —31,
 31 $CCpqCCCprCsCCqrs.$
 20 $p/Cpq, q/CCprs, r/CCqrs$ *C31—32,
 32 $CCCprsCCpqCCqrs.$
- K. Iséki and Y. Arai have proved that Frege axioms of propositional calculus imply $CCNppp$ (for detail, see [1]). Applying a) $CCNppp$, we continue to prove.
- 32 $p/Np, r/p, s/p$ *Ca)—33,
 33 $CCNpqCCqpp.$
 10 $p/CNpq, q/CNqp, r/CCpq$ *C14—C33 $p/q, q/p$ —34,
 34 $CCNpqCCpqq.$
 20 $p/CNpq, q/Cpq, r/q$ *C34—35,

- 35 $CCpqCCNpqq.$
 34 $q/Np *C30 p/Np-36,$
- 36 $CCpNpNp.$
 7 $p/s, q/CpCqr, r/CqCpr *C20-37,$
- 37 $CCsCpCqrCsCqCpr.$
 7 $p/s, q/Cpq, r/CCqrCpr *C10-C37-38,$
- 38 $CCsCpqCsCCqrCpr.$
 37 $p/s, q/Cqr, r/Cpr, s/CsCpq *C38-39,$
- 39 $CCsCpqCCqrCsCpr.$
 39 $p/Cqr, q/Cpr, r/CCNprr, s/Cpq *C10-C35$
 $q/r-40,$
- 40 $CCpqCCqrCCNprr.$
 10 $p/CsCpCqr, q/CsCqCpr, r/CqCsCpr *C37-C20$
 $p/s, r/Cpr-41,$
- 41 $CCsCpCqrCqCsCpr.$
 10 $p/CsCpCqr, q/CqCsCpr, r/CqCpCsr *C41-C37$
 $s/q, p/s, q/p-42,$
- 42 $CCsCpCqrCqCpCsr.$
 42 $p/Cqr, q/CNpr, s/Cpq *C40-43,$
- 43 $CCNprCCqrCCpqr.$
 20 $p/CNpq, q/Cqp, r/p *C33-44,$
- 44 $CCqpCCNpqp.$
 10 $p/CNpNq, q/Cqp, r/CCNpqp *C15-C44-45,$
- 45 $CCNpNqCCNpqp.$

Axiom 1 of (F) and theses 7, 20, 21, 35 are axioms of (H).

Axioms 1, 4 of (F) and theses 10, 20, 28, 36 are axioms of (R).

Theses 22, 29, 43 are axioms of (L_2).

Axioms 1, 2 of (F) and thesis 15 are axioms of (L_3).

Theses 17, 24, 43 are axioms of (S_1) and theses 24, 27, 34 are axioms of (S_2).

Axioms 1, 2 of (F) and thesis 45 are axioms of (M).

Theses 10, 21, a) are axioms of (L_1).

References

- [1] Y. Arai and K. Iséki: On axiom systems of propositional calculi. VII. Proc. Japan Acad., **41**, 667-669 (1965).
- [2] —: On axiom systems of propositional calculi. I. Proc. Japan Acad., **41**, 436-439 (1965).