166. *Notes on Ergodicity and Mixing Property*

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1. In this note we will give the conditions for the validity of ergodicity, mixing property and weakly mixing property in terms of entropy.

Let (X, S_X) be a measurable space where S_X is a σ-field in X, and let γ and μ be two probability measures on S_X. The *entropy rate* $H_\mu(\gamma)$ of γ with respect to μ is defined by

$$H_\mu(\gamma)=\int_X \log \frac{d\gamma}{d\mu}\, d\gamma$$

if γ is absolutely continuous with respect to μ, and otherwise $H_\mu(\gamma)=+\infty$, where $\frac{d\gamma}{d\mu}$ is a Radon-Nikodym density function of γ with respect to μ.*)

Proposition 1. Let μ and γ_t $(0\leqslant t\leqslant +\infty)$ be probability measures on S_X. Suppose that $\gamma_t \leqslant c\mu$ on S_X for any t, where c is a constant $\geqslant 1$. Then

$$\lim_{t\to\infty} \gamma_t(E)=\mu(E)$$

uniformly for $E\in S_X$ if, and only if,

$$\lim_{t\to\infty} H_\mu(\gamma_t)=0 .$$

Proof. Note that $\frac{d\gamma_t}{d\mu}$ are uniformly bounded and that the "only if" assertion is equivalent to that $\frac{d\gamma_t}{d\mu}$ converges to 1 in the L_1-mean (with respect to μ).

Now we prove the "only if" part. Since $\frac{d\gamma_t}{d\mu}$ converges to 1 in probability and

$$|\, x \log x \,|\leqslant|\, x-1\,|+\frac{1}{2}(x-1)^2$$

for any $x\geqslant 0$, so $\frac{d\gamma_t}{d\mu}\log\frac{d\gamma_t}{d\mu}$ converges to 0 in probability. Therefore, since $\frac{d\gamma_t}{d\mu}\log\frac{d\gamma_t}{d\mu}$ are uniformly bounded,

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*) Cf. Prinsker, M. S., Information and information stability of random variables and processes, English edition. translated by A. Feinstein (1964).

$$\lim_{t\to\infty}\int_X \frac{d\gamma_t}{d\mu}\log\frac{d\gamma_t}{d\mu}\,d\mu=0.$$

We prove next the "if" part. Since

$$x\log x\geqslant(x-1)+\frac{1}{2c}(x-1)^2$$

for any x with $0\leqslant x\leqslant c$,

$$\int_X \frac{d\gamma_t}{d\mu}\log\frac{d\gamma_t}{d\mu}\,d\mu\geqslant\frac{1}{2c}\int_X\Big(\frac{d\gamma_t}{d\mu}-1\Big)^2 d\mu\geqslant 0\ .$$

Hence $\frac{d\gamma_t}{d\mu}$ converges to 1 in the L_2-mean and so does in the L_1-mean.

2. Let $(X(0), S_{X(0)})$ be a measurable space, and $(X, S_X)=\bigotimes_{t\geq 0}(X(t), S_{X(t)})$, where $(X(t), S_{X(t)})=(X(0), S_{X(0)})$ for any $t\geqslant 0$. Given a probability measure μ on S_X, we call $\psi=\{\psi_t, t\geqslant 0\}$ a *semi-flow on* (X, S_X, μ) if ψ_t is an endomorphism on (X, S_X, μ) for each t, and ψ a semi-group. We will consider only *measurable* semi-flows, and so the word "measurable" will be omitted in the sequel. For each t, we define a probability measure $\bar{\gamma}_t$ on $S_X\otimes S_X$ by

$$\bar{\gamma}_t(E\otimes F)=\frac{1}{t}\int_0^t \mu(\psi_s^{-1}E\cap F)ds$$

for any $E, F\in S_X$. Let Θ be the class of all finite S_X-partitions of X. For each $\theta\in\Theta$, let μ_θ be the restriction of μ into the σ-field generated by θ and, for each pair $\theta, \theta'\in\Theta$, $\bar{\gamma}_t^{\theta,\theta'}$ the restriction of $\bar{\gamma}_t$ into the σ-field $S(\theta, \theta')$ generated by the class $\{E\otimes F\colon E\in\theta, F\in\theta'\}$.

A semi-flow ψ is called *ergodic* if

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mu(\psi_s^{-1}E\cap F)ds=\mu(E)\mu(F)$$

for any $E, F\in S_x$. Now, we introduce following quantities: for each pair $\theta, \theta'\in\Theta$ and each t,

$$\bar{I}_t(\theta, \theta')=\sum_{E\in\theta, F\in\theta'}\bar{\gamma}_t(E\otimes F)\log\frac{\bar{\gamma}_t(E\otimes F)}{\mu(E)\mu(F)}\ ,$$

$$\bar{H}_t(\theta, \theta')=-\sum_{E\in\theta, F\in\theta'}\bar{\gamma}_t(E\otimes F)\log\bar{\gamma}_t(E\otimes F)\ ,$$

and

$$H(\theta, \theta')=-\sum_{E\in\theta}\mu(E)\log\mu(E)-\sum_{F\in\theta'}\mu(F)\log\mu(F)\ .$$

Proposition 2. Let ψ be a semi-flow. Then the following three assertions are mutually equivalent:

(1) ψ is ergodic.

(2) $\lim_{t\to\infty}\bar{I}_t(\theta, \theta')=0$ for any pair $\theta, \theta'\in\Theta$.

(3) $\lim_{t\to\infty}\bar{H}_t(\theta, \theta')=H(\theta, \theta')$ for any pair $\theta, \theta'\in\Theta$.

Proof. (1)$\Leftrightarrow$(2): The semi-flow is ergodic if, and only if,

$$\lim_{t\to\infty}\bar{\gamma}_t^{\theta,\theta'}(M)=\mu_\theta\otimes\mu_{\theta'}(M)$$

for any $M \in S(\theta, \theta')$ and any pair $\theta, \theta' \in \Theta$. This convergence is uniform for $M \in S(\theta, \theta')$, and

$$\frac{d\bar{\gamma}_t^{\theta,\theta'}}{d\mu_\theta \otimes \mu_{\theta'}} \leqslant \max_{\substack{F \in \theta \\ \mu(F) \neq 0}} \frac{1}{\mu(F)},$$

$$\bar{I}_t(\theta, \theta') = H_{\mu_\theta \otimes \mu_{\theta'}}(\bar{\gamma}_t^{\theta,\theta'})$$

and so, by Prop. 1, (1) and (2) are mutually equivalent.

(2)$\Leftrightarrow$(3): This mutual implication holds trivially, since

$$\bar{I}_t(\theta, \theta') = -\bar{H}_t(\theta, \theta') + H(\theta, \theta')$$

for any θ, θ' and t.

A semi-flow ψ is called *mixing* if

$$\lim_{t \to \infty} \mu(\psi_t^{-1} E \cap F) = \mu(E)\mu(F)$$

for any $E, F \in S_X$. We define a probability measure γ_t on $S_X \otimes S_X$ for each t by

$$\gamma_t(E \otimes F) = \mu(\psi_t^{-1} E \cap F)$$

for any $E, F \in S_X$, and let $\gamma_t^{\theta\,\theta'}$ be the restriction of γ_t into $S(\theta, \theta')$. We introduce moreover following quantities: for each pair θ, θ' and each t,

$$I_t(\theta, \theta') = \sum_{E \in \theta, F \in \theta'} \gamma_t(E \otimes F) \log \frac{\gamma_t(E \otimes F)}{\mu(E)\mu(F)},$$

and

$$H_t(\theta, \theta') = -\sum_{E \in \theta, F \in \theta'} \gamma_t(E \otimes F) \log \gamma_t(E \otimes F).$$

Proposition 3. Let ψ be a semi-flow. Then the following three assertions are mutually equivalent:

(1) ψ is mixing.

(2) $\lim_{t \to \infty} I_t(\theta, \theta') = 0$ for any pair $\theta, \theta' \in \Theta$.

(3) $\lim_{t \to \infty} H_t(\theta, \theta') = H(\theta, \theta')$ for any pair $\theta, \theta' \in \Theta$.

Proof. The proof of Prop. 2 remains valid if therein $\bar{\gamma}_t$, $\bar{I}_t$, and $\bar{H}_t$ are replaced by γ_t, I_t, and H_t, respectively.

A semi-flow ψ is called *weakly mixing* if

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t (\mu(\psi_s^{-1} E \cap F) - \mu(E)\mu(F))^2 ds = 0$$

for each $E, F \in S_X$.

Proposition 4. Let ψ be a semi-flow. Then the following three assertions are mutually equivalent:

(1) ψ is weakly mixing.

(2) $\lim_{t \to \infty} \frac{1}{t} \int_0^t I_s(\theta, \theta') ds = 0$ for any $\theta, \theta' \in \Theta$.

(3) $\lim_{t \to \infty} \frac{1}{t} \int_0^t H_s(\theta, \theta') ds = H(\theta, \theta')$ for any $\theta, \theta' \in \Theta$.

Proof. (1)$\Leftrightarrow$(2): For any θ, θ' and t,

$$\frac{1}{2c}\sum_{\substack{E\in\theta, F\in\theta'\\ \mu(E)\mu(F)\neq 0}}\frac{1}{\mu(E)\mu(F)}\left(\frac{1}{t}\int_0^t(\mu(\psi_s^{-1}E\cap F)-\mu(E)\mu(F))^2ds\right)$$
$$\leqslant \frac{1}{t}\int_0^t I_s(\theta, \theta')ds$$
$$\leqslant \sum_{\substack{E\in\theta, F\in\theta'\\ \mu(E)\mu(F)\neq 0}}\frac{1}{\mu(E)\mu(F)}\left(\frac{1}{t}\int_0^t(\mu(\psi_s^{-1}E\cap F)-\mu(E)\mu(F))^2ds\right),$$

where $c=\max\limits_{\substack{F\in\theta'\\ \mu(F)\neq 0}}\frac{1}{\mu(F)}$. Therefore (1) and (2) are equivalent.

(2)$\Leftrightarrow$(3): This mutual implication is trivial, since

$$\frac{1}{t}\int_0^t I_s(\theta, \theta')ds=-\frac{1}{t}\int_0^t H_s(\theta, \theta')ds+H(\theta, \theta').$$

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