165. *On the Isomorphism Problem of Certain Semigroups Constructed from Indexed Groups*

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T. Tamura, in [1], has showed that the cancellative, archimedean, nonpotent, commutative semigroup S can be constructed from the indexed group G with index I, defining a product in $S=N_0\times G$, where N_0 is the set of all non-negative integers, by $(m, x)(n, y)=(m+n+I(x, y), xy)$ and proposed a problem that under what condition, is S constructed from G with I isomorphic upon S' from G' with I'? In this paper, we shall give a solution without proofs for the above.

1. For any element a of an indexed group G with an index I and any integers r and s we define $\rho_r^s(a)$ as follows:

$$\begin{aligned}\rho_r^s(a)&=\sum_{i=r}^{s} I(a, a^i) && \text{if } s-r\geqq 0,\\ &=0 && \text{if } s-r=-1,\\ &=-\sum_{i=s+1}^{r-1} I(a, a^i) && \text{if } s-r\leqq -2,\end{aligned}$$

where a^0 means the identity element of G.

Then we get the following lemmas:

Lemma 1. *For any integers r, s, and t it holds that*

$$\rho_r^{s-1}(a)+\rho_s^t(a)=\rho_r^t(a).$$

Therefore, immediately

Lemma 2. *If the order of a is m, then, for any integers r and s*

$$\rho_1^{mr+s}(a)=r\rho_1^m(a)+\rho_1^s(a).$$

Lemma 3. *For any integers r and s*

$$I(a^r, a^s)=\rho_s^{r+s-1}(a)-\rho_1^{r-1}(a)=\rho_r^{r+s-1}(a)-\rho_1^{s-1}(a).$$

From Lemmas 1 and 3

Lemma 4. *For any integer r*

$$\rho_1^r(a^{-1})=(r+1)\rho_{-1}^0(a)-\rho_{-r-1}^0(a).$$

2. Let $S=N_0\times G$ and $S'=N_0\times G'$ be cancellative, archimedean, nonpotent, commutative semigroups constructed from indexed groups G with I and G' with I' respectively. Suppose that S is isomorphic upon S' under φ. Let e and e' be the identity elements of G and G' respectively and put $(0, e)\varphi=(n', e_0')$ and $(0, e')\varphi^{-1}=(n, e_0)$. Since $(0, e')=(n, e_0)\varphi=((0, e_0)(0, e)^n)\varphi$, where we agree that $\alpha\beta^0$ means α for every $\alpha, \beta\in S$, we get

Lemma 5. $nn'=0$.

We define the symbol $\rho_r'^s(a'), a' \in G'$ as follows:

$$\begin{aligned} \rho_r'^s(a') &= \sum_{i=r}^{s} I'(a', a'^i) && \text{if } s-r \geqq 0, \\ &= 0 && \text{if } s-r=-1, \\ &= -\sum_{i=s+1}^{r-1} I'(a', a'^i) && \text{if } s-r \leqq -2. \end{aligned}$$

Using, then, Lemmas 1, 3, and 4 we get

Lemma 6. *Let $(p, e_0^k x)$ and $(p', e_0'^{k'} y')$ be any elements of S and S' respectively. If $(0, x)\varphi=(r, x')$ and $(0, y')\varphi^{-1}=(r', y)$, then*
(i) $(p, e_0^k x)\varphi=(r+k+n't+\rho_1'^{t-1}(e_0')+I'(e_0'^t, x'), e_0'^t x')$, $t=p-nk-\rho_1^{k-1}(e_0)-I(e_0^k, x)$,
(ii) $(p', e_0'^{k'} y')\varphi^{-1}=(r'+k'+nt'+\rho_1^{t'-1}(e_0)+I(e_0^{t'}, y), e_0^{t'} y)$, $t'=p'-n'k'-\rho_1'^{k'-1}(e_0')-I'(e_0'^{k'}, y')$.

Lemma 7. *If h and h' are the orders of e_0 and e_0' respectively, then it holds that $h=h'n'+\rho_1'^{h'}(e_0')$ and $h'=hn+\rho_1^h(e_0)$, where $h=0$ means that the order of e_0 is infinite and $h'=0$ does so.*

Since $S\varphi=S'$, by Lemma 6, we get

Lemma 8. *For any integer s and for any $x \in G$ and $y' \in G'$ it holds that*
(i) $r+s+n't(s)+\rho_1'^{t(s)-1}(e_0')+I'(e_0'^{t(s)}, x') \geqq 0, t(s)=-ns-\rho_1^{s-1}(e_0)-I(e_0^s, x)$ *and*
(ii) $r'+s+nt'(s)+\rho_1^{t'(s)-1}(e_0)+I(e_0^{t'(s)}, y) \geqq 0, t'(s)=-n's-\rho_1'^{s-1}(e_0')-I'(e_0'^s, y')$, *where* $(0, x)\varphi=(r, x'), (0, y')\varphi^{-1}=(r', y)$.

Lemma 8 is equivalent to the following

Lemma 9. *For any integer s and for any $x \in G$ and $y' \in G'$ it holds that*
(i) $r+\min\left\{0, \min_{1\leqq s}\left\{s+n't(s)-\sum_{i=t(s)}^{-1} I'(e_0', e_0'^i x')\right\}, \min_{1\leqq s}\left\{-s+n't(-s)+\sum_{i=0}^{t(-s)-1} I'(e_0', e_0'^i x')\right\}\right\} \geqq 0$ *and*
(ii) $r'+\min\left\{0, \min_{1\leqq s}\left\{s+nt'(s)-\sum_{i=t'(s)}^{-1} I(e_0, e_0^i y)\right\}, \min_{1\leqq s}\left\{-s+nt'(-s)+\sum_{i=0}^{t'(-s)-1} I(e_0, e_0^i y)\right\}\right\} \geqq 0$,
where $(0, x)\varphi=(r, x'), (0, y')\varphi^{-1}=(r', y)$.

We note here that for $s \geqq 1$ it holds $t(s) \leqq 0$, $t(-s) \geqq 0$, $t'(s) \leqq 0$ and $t'(-s) \geqq 0$ and we agree that both the symbols $\sum_{i=m}^{m-1} I'(e_0', e_0'^i x')$ and $\sum_{i=m}^{m-1} I(e_0, e_0^i y)$ mean 0. And if the orders of e_0 and e_0' are $h(\neq 0)$ and $h'(\neq 0)$ respectively, we may restrict the s in (i) of Lemmas 8, 9 as $0 \leqq s < h$ and the s in (ii) as $0 \leqq s < h'$ and may cancel the terms

$$\min_{1\leqq s}\left\{-s+n't(-s)+\sum_{i=0}^{t(-s)-1} I'(e_0', e_0'^i x')\right\}$$

in (i),

$$\min_{1\leq s}\left\{-s+nt'(-s)+\sum_{i=0}^{t'(-s)-1} I(e_0, e_0^i y)\right\}$$

in (ii) of Lemma 9.

Moreover we have the following lemmas:

Lemma 10. *If* $(0, x')\varphi^{-1}=(r, x)$, $r\neq 0$, *then* $(0, e_0'^{s-r}x')\varphi^{-1}=(s, x)$ *for any integer* s *such that* $0\leqq s\leqq r$. *And if* $(0, y)\varphi=(r', y')$, $r'\neq 0$, *then* $(0, e_0^{s'-r'}y)\varphi=(s', y')$ *for any integer* s' *such that* $0\leqq s'\leqq r'$.

Lemma 11. *Let* $(0, x)\varphi=(0, x')$ *and* $(0, y)\varphi=(0, y')$. *If* $xy=e_0^k z$ *and* $(0, z)\varphi=(0, z')$, *then* $x'y'=e_0'^{k'}z'$ *and* $I'(x', y')=n'k'+k+\rho'^0_{-k'}(e_0')-I'(e_0'^{-k'}, x'y')$, *where* $k'=-nk-\rho^0_{-k}(e_0)+I(x, y)+I(e_0^{-k}, xy)$.

Lemma 12. *If* $(0, e_0^i)\varphi=(0, e_0'^{i'})$, *then* $i=-n'i'+\rho'^0_{i'}(e_0')$ *and* $i'=-ni+\rho^0_i(e_0)$.

From Lemmas 6 and 10 we can easily see that it is possible to choose a representative system Γ of the cosets of the cyclic subgroup $[e_0]$ generated by e_0 in G and a representative system Γ' of the cosets of $[e_0']$ in G' satisfying the following: for any $x_\alpha\in\Gamma$, there exists $x_\alpha'\in\Gamma'$ such that $(0, x_\alpha)\varphi=(0, x_\alpha')$.

Now, for these Γ and Γ', we define a mapping ψ of $G/[e_0]$ into $G'/[e_0']$ as follows: for $x_\alpha\in\Gamma$ and for $x_\alpha'\in\Gamma'$ such that $(0, x_\alpha)\varphi=(0, x_\alpha')$

$$\psi\colon [e_0]x_\alpha\to[e_0']x_\alpha'.$$

Using Lemma 11, we see that ψ is an isomorphism of $G/[e_0]$ upon $G'/[e_0']$. Therefore

Lemma 13. $G/[e_0]$ *is isomorphic upon* $G'/[e_0']$.

3. Summarizing the above lemmas, we get the following:

Theorem. $S=N_0\times G$ *is isomorphic upon* $S'=N_0\times G'$ *if and only if there exist cyclic subgroups* $[e_0]$ *of* G *and* $[e_0']$ *of* G' *such that* $G/[e_0]$ *is isomorphic upon* $G'/[e_0']$ *(under* ψ*) and there exist representative systems* $\Gamma=\{x_\alpha\}$ *of the cosets of* $[e_0]$ *in* G *and* $\Gamma'=\{x_\alpha'\}$ *of the cosets of* $[e_0']$ *in* G' *satisfying*

(1) *for any* $x_\alpha, x_\beta\in\Gamma$, *if* $x_\alpha x_\beta=e_0^l x_\gamma$, $x_\gamma\in\Gamma$, *then* $x_\alpha\tau\cdot x_\beta\tau=e_0'^{l'}\cdot x_\gamma\tau$ *and* $I'(x_\alpha\tau, x_\beta\tau)=n'l'+l+\rho'^0_{-l'}(e_0')-I'(e_0'^{-l'}, x_\alpha\tau\cdot x_\beta\tau)$, $l'=-nl+I(x_\alpha, x_\beta)-\rho^0_{-l}(e_0)+I(e_0^{-l}, x_\alpha x_\beta)$,

(2) *for any integer* s *and for any* $x_\alpha\in\Gamma$ *and* $x_\beta'\in\Gamma'$

(i) $s+n't(s)+\rho_1'^{t(s)-1}(e_0')+I'(e_0'^{t(s)}, x_\alpha\tau)\geqq 0$, $t(s)=-ns-\rho_1^{s-1}(e_0)-I(e_0^s, x_\alpha)$ *and*

(ii) $s+nt'(s)+\rho_1^{t'(s)-1}(e_0)+I(e_0^{t'(s)}, x_\beta'\tau^{-1})\geqq 0$, $t'(s)=-n's-\rho_1'^{s-1}(e_0')-I'(e_0'^s, x_\beta')$,

where τ *is a mapping of* Γ *onto* Γ' *such that*

$$\tau\colon x_\alpha\to x_\alpha' \quad \textit{if } ([e_0]x_\alpha)\psi=[e_0']x_\alpha'$$

and n, n' *are non-negative integers such that* $nn'=0$, $n'i'=-i+\rho'^0_{i'}(e_0')$, $ni=-i'+\rho^0_i(e_0)$ *for* $e_0^i\tau=e_0'^{i'}$, $e_0^i\in\Gamma$.

References

[1] T. Tamura: Commutative nonpotent archimedean semigroup with cancellation law, I. Jour. of Gakugei, Tokushima Univ., **8**, 5-11 (1957).

[2] A. H. Clifford and G. B. Preston: The algebraic theory of semigroups, I. Amer. Math. Soc., Providence, R. I. (1961).