201. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XIX

By Sakuji INOUE

Faculty of Education, Kumamoto University

(Comm. by Kinjirô KUNUGI, M.J.A., Dec. 13, 1965)

We next discuss the case where the ordinary part of $T(\lambda)$ is a polynomial of degree d.

Theorem 52. Let $T(\lambda)$ and σ be the same notations as before; let the ordinary part $R(\lambda)$ of $T(\lambda)$ be a polynomial in λ of degree d; let c be any finite complex number; let $n_d(\rho, c)$ denote the number of all the c-points, with due count of multiplicity, of $T(\lambda)$ in the domain $\Delta_\rho\{\lambda : \rho<|\lambda|<\infty\}$ with $\sigma<\rho<\infty$; let e_d denote the coefficient of λ^d in the expansion of $R(\lambda)$; let

$$N_d(\rho, c)=\int_\rho^\infty \frac{n_d(r, c)}{r} dr \quad (\sigma<\rho<\infty);$$

let

$$m_d(\rho, c)=\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{[T(\rho e^{-it}), c]} dt \quad (\sigma<\rho<\infty);$$

and let

$$m_d(\infty, c)=\lim_{\rho\to\infty} m_d(\rho, c)(=\log\sqrt{1+|c|^2}).$$

Then the equality

$$N_d(\rho, c)+m_d(\rho, c)-m_d(\infty, c)+\log|e_d|=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\sqrt{1+|T(\rho e^{-it})|^2}}{\rho^d} dt$$

holds for every finite value c and every ρ with $\sigma<\rho<\infty$; and both the left and right sides of this equality converge to $\log|e_d|$ as ρ becomes infinite.

Proof. Suppose that $R(\lambda)=\sum_{\mu=0}^{d} e_\mu\lambda^\mu$, $(e_d\neq 0)$, and consider the function $g(\lambda)$ defined by

$$g(\lambda)=\begin{cases}\lambda^d\left[T\left(\frac{1}{\lambda}\right)-c\right] & \left(0<|\lambda|\leqq\frac{1}{\rho},\ \sigma<\rho<\infty\right)\\ e_d & (\lambda=0).\end{cases}$$

Then $g(\lambda)=\sum_{\mu=0}^{d} e_\mu\lambda^{d-\mu}+\sum_{\mu=1}^{\infty} C_{-\mu}\lambda^{d+\mu}-c\lambda^d$ where $C_{-1}, C_{-2}, C_{-3}, \cdots$ are the coefficients stated at the beginning of the proof of Theorem 47, and $g(\lambda)$ is regular in the closed domain $\left\{\lambda : 0\leqq|\lambda|\leqq\frac{1}{\rho}\right\}$. If we now denote all the zeros, repeated according to the respective orders, of $g(\lambda)$ in the domain $\left\{\lambda : 0<|\lambda|<\frac{1}{\rho}\right\}$ by $a_1, a_2, \cdots, a_{n(\rho)}$, then all the c-points, repeated according to the respective orders, of $T(\lambda)$ in the domain $\{\lambda : \rho<|\lambda|<\infty\}$ are given by $a_1^{-1}, a_2^{-1}, \cdots, a_{n(\rho)}^{-1}$. By making use of Jensen's theorem for $g(\lambda)$, we have

$$\log|g(0)|+\log\frac{1}{|a_1a_2\cdots a_{n(\rho)}|\rho^{n(\rho)}}=\frac{1}{2\pi}\int_0^{2\pi}\log\left|g\left(\frac{1}{\rho}e^{it}\right)\right|dt\ (\sigma<\rho<\infty),$$

where it is easily verified that

$$N_d(\rho, c)=\int_\rho^\infty\frac{n_d(r, c)}{r}dr$$
$$=\log\frac{|a_1^{-1}a_2^{-1}\cdots a_{n(\rho)}^{-1}|}{\rho^{n(\rho)}}.$$

Thus we obtain

$$\log|e_d|+N_d(\rho, c)=\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{T(\rho e^{-it})-c}{(\rho e^{-it})^d}\right|dt\ (\sigma<\rho<\infty)$$

and there is no difficulty in showing from this result that the desired equality in the statement of the present theorem holds for every finite value c every ρ with $\sigma<\rho<\infty$. Since, in addition,

$$\frac{\sqrt{1+|T(\rho e^{-it})|^2}}{\rho^d}=\sqrt{\frac{1}{\rho^{2d}}+\left|\frac{T(\rho e^{-it})}{(\rho e^{-it})^d}\right|^2}\to|e_d|\qquad(\rho\to\infty),$$

it is at once obvious that both the left and right sides of that desired equality converge to $\log|e_d|$ as ρ becomes infinite.

The proof of the theorem is thus complete.

Theorem 53. Let $T(\lambda)$ and σ be the same notations as before; let the ordinary part of $T(\lambda)$ be a polynomial in λ of degree d; and let

$$T_d(\rho)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\sqrt{1+|T(\rho e^{-it})|^2}}{\rho^d}dt.$$

Then $T_d(\rho)$ is not only a monotone decreasing function of ρ but also a convex function of $\log\rho$ in the interval $\sigma<\rho<\infty$.

Proof. By virtue of Theorem 52, we have

$$T_d(\rho)=\frac{1}{\pi}\iint_A N_d(\rho, c)d\omega(c)+\frac{1}{\pi}\iint_A m_d(\rho, c)d\omega(c)$$
$$-\frac{1}{\pi}\iint_A\log\sqrt{1+|c|^2}\,d\omega(c)+\log|e_d|,$$

where $\frac{1}{\pi}\iint_A m_d(\rho, c)d\omega(c)$ is a finite positive constant irrespective of ρ and T, and so also is $\frac{1}{\pi}\iint_A\log\sqrt{1+|c|^2}\,d\omega(c)$. Putting

$$S_d(\rho)=\frac{1}{\pi}\iint_A n_d(\rho, c)d\omega(c),$$

we obtain therefore $T'_d(\rho)=-\dfrac{S_d(\rho)}{\rho}<0$ for every ρ with $\sigma<\rho<\infty$. Since, in addition, $T_d(\rho)\to\log|e_a|$ $(\rho\to\infty)$, we have

$$T_d(\rho)=\int_\rho^\infty \frac{S_d(\rho)}{\rho}\,d\rho+\log|e_a|$$

and $\dfrac{d^2T_d(\rho)}{d(\log\rho)^2}=-S'_d(\rho)\rho$, where $S_d(\rho)$ is a monotone decreasing function of ρ in the open interval (σ,∞) as will be seen from the definition of $n_d(\rho,c)$. Hence $\dfrac{d^2T_d(\rho)}{d(\log\rho)^2}\geqq 0$ for every $\rho\in(\sigma,\infty)$.

With these results, the theorem has been proved.

We can decide by $T(\lambda)$ itself whether its ordinary part is a constant or a polynomial.

Remark A. A necessary and sufficient condition that the ordinary part $R(\lambda)$ of the function $T(\lambda)$ treated above be a constant ξ (inclusive of 0) is that the equality

$$\frac{1}{2\pi i}\int_{|\lambda|=\rho}\frac{T(\lambda)}{\lambda-z}\,d\lambda=\xi \qquad (\sigma<\rho<\infty)$$

be valid for every z inside the circle $|\lambda|=\rho$, positively oriented [cf. Proc. Japan Acad., 40 (7), 492–497 (1964)].

Remark B. A necessary and sufficient condition that $R(\lambda)$ be a polynomial of degree d is that $\dfrac{T(\lambda)}{\lambda^d}$ tend to a non-zero finite value when $|\lambda|\to\infty$ [cf. Proc. Japan Acad., 40 (8), 654–659 (1964)].

Theorem 54. If, in Theorem 53, for any large positive number G there exist a positive constant ρ_G in a bounded open interval (σ, l), $(\sigma<l<\infty)$, and a set $A_{\theta(\rho_G)}$, with positive measure m_G, of angles θ such that the inequality $|T(\rho_G e^{-i\theta})|>G$ holds for every $\theta\in A_{\theta(\rho_G)}$ and that $\inf_G m_G>0$, then, for uncountably many complex numbers $\{c\}$ chosen suitably, $T(\lambda)$ has a denumerably infinite number of c-points $b_\mu^{(c)}(\mu=1,2,3,\cdots)$, repeated according to the respective orders, in the domain $\Delta_\sigma\{\lambda:\sigma<|\lambda|<\infty\}$ such that any accumulation point of them lies on the circle $|\lambda|=\sigma$ and that the positive series $\sum_{\mu=1}^\infty(|b_\mu^{(c)}|-\sigma)$ is divergent.

Proof. Since, as we have already proved in Theorem 44, the maximum modulus $M_T(\rho)$ of $T(\lambda)$ on the circle $|\lambda|=\rho$ with $\sigma<\rho<\infty$ becomes infinite as ρ tends to σ, and since $T(\lambda)$ is regular in the domain Δ_σ defined above, there exist for any large $G(>0)$ a positive constant ρ_G in (σ, l) and a set $A_{\theta(\rho_G)}$, with positive measure m_G, of angles θ such that the inequality

$$\frac{1}{2\pi}\int_0^{2\pi} \log \sqrt{1+|T(\rho_G e^{-it})|^2}\, dt > \frac{m_G}{2\pi} \log \sqrt{1+G^2}$$

holds, however large G may be; and moreover, by hypothesis, there exists some positive constant K such that $m_G \geqq K$, no matter how large G may be. As a result, we find that $T_d(\rho) \to \infty$ $(\rho \to \sigma)$. Applying this result to the equality established at the beginning of the proof of Theorem 53, it is easily showm that

$$\frac{1}{\pi}\iint_A N_d(\rho, c) d\omega(c) \to \infty \qquad (\rho \to \sigma).$$

In consequence, there exists at least one finite value c such that

$$N_d(\rho, c) = \log \frac{|b_1^{(c)} b_2^{(c)} \cdots b_{n(\rho)}^{(c)}|}{\rho^{n(\rho)}}$$
$$= \log \prod_{\mu=1}^{n(\rho)} \left(1 + \frac{|b_\mu^{(c)}| - \rho}{\rho}\right) \to \infty \qquad (\rho \to \sigma),$$

where $b_1^{(c)}, b_2^{(c)}, \cdots,$ and $b_{n(\rho)}^{(c)}$ denote all the c-points, repeated according to the respective orders, of $T(\lambda)$ in the domain $\varDelta_\rho\{\lambda : \rho < |\lambda| < \infty\}$ with $\sigma < \rho < \infty$. Since, in addition, we can choose a system of open domains $\{\mathfrak{D}_\mu\}_{\mu=1,2,3,\cdots}$ with $b_{\mu+1}^{(c)} \in \mathfrak{D}_{\mu+1} \subset \mathfrak{D}_\mu$ such that $\mathfrak{D}_\mu$ does not converge to a point when $\mu \to \infty$, the statement of the present theorem follows at once from the result just established and the regularity of $T(\lambda)$ in $\varDelta_\sigma$.

Theorem 55. Even if the ordinary part of $T(\lambda)$ is a constant (inclusive of 0), the same result as that stated in Theorem 54 is also valid under the above-mentioned hypothesis concerning m_G.

Proof. By making use of the same reasoning as that applied to prove Theorem 54 and of Theorems 43, 46, 47, 49, and 50, we can easily show the validity of the present theorem.