190. Doubly Extended Geometries by Non-Connection Methods

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The most important problem of geometry seems to be a generalization of the "Erlanger Programm" of Felix Klein (1872) to the case of differentiable manifolds. E. Cartan consacrated almost whole of his life to investigations along the line of Lie groups to this purpose and gave a few local connection geometries parallel to the classical geometries in the sense of the "Erlanger Programm" but without arriving at his own main goal. S. S. Chern [9] and C. Ehresmann [10, 11], A. Lichnerowicz [12] and T. Ōtsuki [13–16] have attempted to establish a *global* theory of connections leading to the cross sections of the principal fibre bundles introducing connections in them.

In a series of previous papers of the present author ([1–8], [18–25]), he has established *extended geometries* corresponding to the 22 branches shown in the system on p. 247 of [22] (to be referred to by $*$). In case of the extended affine geometry (and for other branches of geometry *Mutatis mutandis*), he has discoverd the II-*geodesic curves corresponding to* $\omega^l_\mu(x)$:

(1) $$d(\omega^l/dt)/dt \equiv \omega^l_\lambda(\ddot{x}^\lambda + \Lambda^\lambda_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu) = 0, \qquad (\omega^l = \omega^l_\mu(x)dx^\mu),$$

where (x^λ) are the local coordinates in a subset U_α of a differentiable manifold $M = \bigcup_\alpha U_\alpha$, $|\omega^l_\mu| \neq 0$ in M, $(\Omega^\lambda_l\omega^l_\mu = \delta^\lambda_\mu \Longleftrightarrow \Omega^\lambda_k\omega^h_\lambda = \delta^h_k$; $d\omega^l_\mu - \Lambda^\lambda_{\mu\nu}\omega^l_\lambda dx^\nu = 0$, $\Lambda^\lambda_{\mu\nu} = \Omega^\lambda_l\partial_\nu\omega^l_\mu \equiv -\omega^l_\mu\partial_\nu\Omega^\lambda_l$, $(\lambda, \mu, \cdots; l, h, \cdots = 1, 2, \cdots, n))$;

(2) $$d\xi^l = a^l dt = \omega^l, \ \xi^l = a^l t + c^l, \quad (a^l = \text{const.}, \ c^l = \text{const.})$$

and adopted the curve $\xi^l = a^l t + c^l$ as the ξ^l-axis. As the equation $\xi^l = a^l t + c^l$ tells us, the II-geodesic curves behave as for meet and join like straight lines. From (2), it follows that $dx^\lambda/dt = a^m\Omega^\lambda_m$ along the II-geodesic curve corresponding to $\omega^l_\mu(x)$. (ξ^l) were called the II-*geodesic parallel coordinates*. When ξ^l and $\bar{\xi}^l$ stands for x^λ and ξ^l respectively, we had come to consider

(3) $d\bar{\xi}^l = a^l_m(\xi)d\xi^m$, $(|a^l_m| \neq 0)$, (4) $\bar{\xi}^l = a^l_m(\xi)\xi^m + a^l_0$, $(a^l_0 = \text{const.})$.

The conditions for the correspondence of $d^2\bar{\xi}^l/dt^2 = 0$ and $d^2\xi^l/dt^2 = 0$:

(5) $$da^l_m(\xi)d\xi^m = 0, \quad da^l_m(\xi)\xi^m = 0.$$

The totality of the transformations of the type (4) forms an *extended affine transformation group. All the extended geometries tabulated in $*$ are realized in the differentiable manifold $M = \bigcup_\alpha U_\alpha$ and belong to the "Erlanger Programm" of F. Klein, so that connections* *become not necessarily indispensable* ([1–8], [18–25]).

We are now in the situation *to extended all such geometries doubly to* the case:

(6) $\omega^l=\omega^l_\mu(x, \dot{x}, \cdots, \overset{(m)}{x})dx^\mu$, $(|\omega^l_\mu|\neq 0$ in M; $\dot{x}=dx/dt$, etc.),

including the geometries of Finsler-Craig-Synge-Kawaguchi spaces [26–30]. In this note, it is commenced with the cases of the *doubly extended affine* and the *doubly extended Euclidean geometries.*

1. High-orderly line-elemented II-geodesic curves. Set

(1.1) $\omega^l\overset{\text{def}}{=}\omega^l_\mu(x, \dot{x}, \cdots, \overset{(m)}{x})dx^\mu$, $(l, m, \cdots; \lambda, \mu, \cdots =1, 2, \cdots, n)$,

where $\dot{x}=dx/dt$, etc. and the 1-forms ω^l are assumed to be *not exact* in general and to be linearly independent, so that the condition

(1.2) $|\omega^l_\mu(x, \dot{x}, \cdots, \overset{(m)}{x})|\neq 0$ in a differentiable manifold M

is satisfied. Since (1.1) is written in an *invariant form,* ω^l are *global* in $M=\bigcup_\alpha U_\alpha$.

For the system $\omega^l_\mu(x, \dot{x}, \cdots, \overset{(m)}{x})$, we introduce $\Omega^\lambda_l(x, \dot{x}, \cdots, \overset{(m)}{x})$ by the condition:

(1.3) $$\Omega^\lambda_l\omega^l_\mu=\delta^\lambda_\mu \Longleftrightarrow \Omega^\lambda_k\omega^h_\lambda=\delta^h_k,$$

where the δ's are Kronecker deltas. We define the connection parameters $\Lambda^\lambda_{\mu\nu}(x, \dot{x}, \cdots, \overset{(m)}{x})$ of teleparallelism for ω^l_μ and Ω^λ_l by

(1.4) $$d\omega^l_\mu-\Lambda^\lambda_{\mu\nu}\omega^l_\lambda dx^\nu=0, \quad \text{(cf. (1.6))}.$$

Consider a parametrized curve $x^\lambda=x^\lambda(t)$.

A straight forward calculation shows us the identity (cf. (1)):

(1.5) $$d(\omega^l/dt)/dt\equiv\omega^l_\lambda(x, \dot{x}, \cdots, \overset{(m)}{x})(\ddot{x}^\lambda+\Lambda^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu),$$

(1.6) $$\Lambda^\lambda_{\mu\nu}\overset{\text{def}}{=}\Omega^\lambda_l(x, \dot{x}, \cdots, \overset{(m)}{x})\left[\frac{\partial}{\partial x^\nu}+\frac{\ddot{x}^\sigma}{\dot{x}^\nu}\frac{\partial}{\partial\dot{x}^\sigma}+\cdots+\frac{\overset{(m+1)}{x^\sigma}}{\dot{x}^\nu}\frac{\partial}{\partial\overset{(m)}{x^\sigma}}\right]\omega^l_\mu(x, \dot{x}, \cdots, \overset{(m)}{x}).$$

From (1.5), we obtain

(1.7) (i) $d(\omega^l/dt)/dt=0$, (*global*). | (ii) $\ddot{x}^\lambda+\Lambda^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=0$, (local).

The differential equations

$$\text{(ii)} \;=\ddot{x}^\lambda+\bar{\Lambda}^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=0, \quad (\bar{\Lambda}^\lambda_{\mu\nu}=(\Lambda^\lambda_{\mu\nu}-\Lambda^\lambda_{\nu\mu})/2)$$

define the *autoparallel curves* of teleparallelism of ω^l_μ and Ω^λ_l.

Indeed, we can easily deduce (1.6) from

$$d\omega^l_\mu-\Lambda^\lambda_{\mu\nu}\omega^l_\lambda dx^\nu=0 \text{ or } d\Omega^\lambda_l+\Lambda^\lambda_{\mu\nu}\Omega^\mu_l dx^\nu=0.$$

The (i) is convenient for the study of the global properties: *The identity*

(1.5)′ $$\Omega^\lambda_l d(\omega^l/dt)/dt\equiv\ddot{x}^\lambda+\Lambda^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

transforms the global path (i) *piece-wise onto the local path* (ii) *by the inverse transformation of* (1.1):

$$(1.1)' \qquad dx^\lambda = \Omega_l^\lambda(x, \dot{x}, \cdots, \overset{(m)}{x})\omega^l.$$

The differential equations are integrated readily:

$$(1.8)\ \omega^l = a^l dt,\ (a^l\text{: const.}), \qquad (1.9)\ \int(\omega^l/dt)dt = a^l t + c^l,\ (c^l\text{: const.}),$$

the integration (1.9) being guided by the simple clear form $a^l dt$ of (1.8). We set $\xi^l = a^l t + c^l$, so that

$$(1.10) \qquad \xi^l = \int(\omega^l/dt)dt = a^l t + c^l.$$

From (1.10), we see that *the curves represented by* (1.7), (i) or *by* (1.10) *behave as for meet and join like straight lines.* We will call these curves *high-orderly line-elemented* II-*geodesic curves.*

The (1.1) *may be rewritten as follows*:

$$(1.13) \qquad d\xi^l = a_\mu^l(x, \dot{x}, \cdots, \overset{(m)}{x})dx^\mu.$$

The first equation (i) of (1.7) may now be rewritten as follows:

$$(1.14) \qquad d^2\xi^l/dt^2 = 0.$$

Multiplying (1.8) with Ω_l^λ, we see that *the relations*

$$(1.15) \qquad dx^\lambda/dt = a^h\Omega_h^\lambda(x, \dot{x}, \cdots, \overset{(m)}{x})$$

hold along the high-orderly line-elemented II-*geodesic line-elements.*

We will call the (ξ^l) the *doubly extended* II-*geodesic parallel coordinates corresponding to* $a_\mu^l(x, \dot{x}, \cdots, \overset{(m)}{x})$ *referred to the high-orderly line-elemented* II-*geodesic coordinate axes.* The (ξ^l) are *global for* $\bigcup_\alpha U_\alpha$.

2. Double extension of the affine transformation group by extending the group parameters doubly to functions of coordinates x^λ and line-elements of higher order. In particular, the (ξ^l) can stand for (x^λ), so that we come to consider

$$(2.1) \qquad d\bar{\xi}^l = a_h^l(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})d\xi^h, \quad (\,|\,a_h^l(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})\,| \neq 0 \text{ in } M)$$

in place of (1.13) for the II-geodesic line-elements of higher order corresponding to $a_h^l(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})$.

In order that the high-orderly line-elemented II-geodesic curves $\xi^l(t)$, $(d^2\xi^l/dt^2 = 0)$ may be transformed by (2.1) into high-orderly line-elemented II-geodesic curves $\bar{\xi}^l(t)$, $(d^2\bar{\xi}^l/dt^2) = 0$ corresponding to $a_h^l(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})$, we must have

$$(2.2) \qquad da_h^l(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})d\xi^h = 0$$

along the high-orderly line-elemented II-*geodesic line-elements.* For, from (2.1), we obtain

$$(2.3) \qquad \ddot{\bar{\xi}}^l = \dot{a}_h^l(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})\dot{\xi}^h + a_h^l(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})\ddot{\xi}^h.$$

Integrating (2.1) along the $\bar{\xi}^l$-axis, we obtain

$$\bar{\xi}^l = a_h^l(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})\xi^h - \int\xi^h(da_h^l(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})/dt)dt.$$

Now

$$\int \xi^h \frac{da^l_h}{dt} dt = \int \frac{da^l_h}{dt} dt \int d\xi^h = \iint \left\{ \frac{da^l_h}{dt} dt d\xi^h \right\} = \text{const.} = a^l_0, \text{ say,}$$

by (2.3). Thus we have

(2.4) $$\bar{\xi}^l = a^l_h(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})\xi^h + a^l_0.$$

We will call the transformation (2.4) *doubly extended affine transformation.*

From (2.2) and (2.4), we see that

(2.5) $$da^l_h(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})\xi^h = 0$$

for the high-orderly line-elemented II-*geodesic line-elements* (cf. (5)).

That *the totality of the doubly extended affine transformations*

(2.6) $$\bar{\xi}^h = a^h_k(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})\xi^k + a^h_0, \quad (a^h_0 = \text{const.}, \ |a^h_k| \neq 0),$$

whose inverse transformations are

(2.7) $$\xi^k = \Omega^k_h(\bar{\xi}, \dot{\bar{\xi}}, \cdots, \overset{(m)}{\bar{\xi}})\bar{\xi}^h + \Omega^k_0, \quad (\Omega^k_0 = \text{const.}, \ |\Omega^k_h| \neq 0),$$

(2.8) $$a^h_k \Omega^l_h = \delta^l_k \Longleftrightarrow a^l_h \Omega^h_k = \delta^l_k,$$

forms a group ($\bar{\mathfrak{G}}$, say), may be proved quite as in p. 65 of [23]. We will call the group $\bar{\mathfrak{G}}$ the *doubly extended affine group.* $\bar{\mathfrak{G}}$ *contains the extended affine group* $\mathfrak{G}$ ([23], p. 65) *as subgroup. The group* $\mathfrak{G}$ *contains the ordinary affine group as subgroup.*

3. Doubly extended equi-affine group. The totality of the elements of the doubly extended affine group such that

(3.1) $$|a^l_h(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})| = 1$$

form a subgroup of $\bar{\mathfrak{G}}$, which we will call the *doubly extended equi-affine group. It contains the extended equi-affine group* ([23], p. 61) *as subgroup.*

4. Another procedure. Another procedure of this note is to start with the fact that there exist in every differentiable manifolds $M = \bigcup_\alpha U_\alpha$ II-geodesic curves (1). For them, (1.8), (1.15), (2.2), (2.3), and (2.5) become respectively to:

(4.1) $\omega^l = \omega^l_\mu(x)dx^\mu = a^l dt$, (4.2) $dx^\lambda/dt = a^h \Omega^\lambda_h(x) = \alpha^\lambda$,

(4.3) $\xi^l = a^l_\mu(x)x^\mu + a^l_0$, (4.4) $da^l_\mu(x)dx^\mu = 0$, (4.5) $da^l_\mu(x)x^\mu = 0$,

provided that (x^λ) *themselves are* II-*geodesic parallel coordinates* corresponding to $a^l_\mu(x)$, [23].

If we utilize such special coordinates (x^λ), then (1.13), (1.15), (2.1), (2.2), (2.3), and (2.5) become respectively to

(4.6) $$\omega^l = d\xi^l = a^l_\mu(x, \alpha, 0, \cdots, 0)dx^\mu,$$

(4.7) $$\dot{x}^\lambda = a^h \Omega^\lambda_h(x, \alpha, \cdots, 0) = \alpha^\lambda,$$

(4.8) $$d\bar{\xi}^l = a^l_h(\xi, \alpha, 0, \cdots, 0)d\xi^h,$$

(4.9) $$\xi^l = a^l_\mu(x, \alpha, 0, \cdots, 0)x^\mu + a^l_0,$$

(4.10) $$da^l_\mu(x, \alpha, 0, \cdots, 0)dx^\mu = 0,$$

(4.11) $$da^l_\mu(x, \alpha, 0, \cdots, 0)x^\mu = 0.$$

5. Doubly extended affine connection. For a given doubly extended local affine connection $\Gamma^{\lambda}_{\mu\nu}(x, \dot{x}, \cdots, \overset{(m)}{x})$, we define a doubly extended global affine connection $\Gamma^{l}_{hk}(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})=\Gamma^{l}_{hk}(\xi, a, \cdots, 0)$ by

(5.1) $\Gamma^{l}_{hk}\Omega^{\lambda}_{l}d\xi^{k}=\Gamma^{\lambda}_{\mu\nu}\Omega^{\mu}_{h}dx^{\nu}+d\Omega^{\lambda}_{h}$, where (5.2) $d\Omega^{\lambda}_{h}=-\Lambda^{\lambda}_{\mu\nu}\Omega^{\mu}_{h}dx^{\nu}$.

Thus we have

(5.3) $\Gamma^{l}_{hk}\Omega^{\lambda}_{l}d\xi^{k}=(\Gamma^{\lambda}_{\mu\nu}-\Lambda^{\lambda}_{\mu\nu})\Omega^{\mu}_{h}dx^{\nu}$, (5.4) $\Gamma^{l}_{hk}\omega^{h}_{\mu}\omega^{k}_{\nu}=\Gamma^{\lambda}_{\mu\nu}-\Lambda^{\lambda}_{\mu\nu}$.

Hence

(5.5) $$\ddot{\xi}^{l}+\Gamma^{l}_{hk}\dot{\xi}^{h}\dot{\xi}^{k}=\ddot{\xi}^{l}+\omega^{l}_{\lambda}(\Gamma^{\lambda}_{\mu\nu}-\Lambda^{\lambda}_{\mu\nu})\dot{x}^{\mu}\dot{x}^{\nu}.$$

Now we have

(5.6) $$\ddot{\xi}^{l}\equiv\omega^{l}_{\lambda}(\ddot{x}^{\lambda}+\Lambda^{\lambda}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}).$$

Adding (5.5) and (5.6) side by side, we obtain

(5.7) $$\ddot{\xi}+\Gamma^{l}_{hk}\dot{\xi}^{h}\dot{\xi}^{k}=\omega^{l}_{\lambda}(\ddot{x}^{\lambda}+\Gamma^{\lambda}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}).$$

6. Doubly extended affine geometry by a non-connection method. We can extend doubly the Friesecke's formula [26]:

In order that $\bar{L}^{\lambda}_{\mu\nu}$ and $L^{\lambda}_{\mu\nu}$ may give one and the same parallelism, it is necessary and sufficient that there exists a non-null vector ψ_{ν}, such that

(6.1) $$\bar{L}^{\lambda}_{\mu\nu}=L^{\lambda}_{\mu\nu}+\delta^{\lambda}_{\mu}\psi_{\nu}+\delta^{\lambda}_{\nu}\psi_{\mu}.$$

Hence (5.4) and (6.1) give

(6.2) $$\Omega^{\lambda}_{l}\omega^{h}_{\mu}\omega^{k}_{\nu}\Gamma^{l}_{hk}=\delta^{\lambda}_{\mu}\psi_{\nu}+\delta^{\lambda}_{\nu}\psi_{\mu}.$$

By contraction $\mu\rightarrow\lambda$, we see that the vector ψ_{ν} exists actually by (5.4) as will be seen as follows:

(6.3) $$(n+1)\psi_{\nu}=\omega^{q}_{\nu}\Gamma^{p}_{pq}=\Gamma^{\lambda}_{\lambda\nu}-\Gamma^{\lambda}_{\nu\lambda}.$$

Hence the

Theorem. *The high-orderly line-elemented* II-*geodesic curves in the large* (1.14) *consists actually of the piece-wise coherence continuation of the local paths* $\ddot{x}^{\lambda}+\Lambda^{\lambda}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}=0$ *by development by the developping factor* $\omega^{l}_{\lambda}(x, \dot{x}, \cdots, \overset{(m)}{x})$.

7. Doubly extended Euclidean geometry by a non-connection method.

(7.1) $$ds^{2}=g_{\mu\nu}(x, \dot{x}, \cdots, \overset{(m)}{x})dx^{\mu}dx^{\nu}$$

is always expressible in the form $ds^{2}=\omega^{l}\omega^{l}$, where ω^{l} is of the type (6). In this case the results of Art. 6 give a doubly extended Euclidean geometry by a non-connection method (cf. [2]).

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