12. A Duality Theorem for Locally Compact Groups. II

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1. Let G be a locally compact group, Ω be the set of all equivalence classes of unitary representations of G. We consider a representative $D=\{U_g^D, \mathfrak{H}^D\}$ of each element in Ω. Denote by $\boldsymbol{T}=\{T(D)\}$ an operator field over Ω, and call $\boldsymbol{T}$ *admissible* when

(1) $T(D)$ is a unitary operator in $\mathfrak{H}^D$ for any D in Ω.

(2) $$U_1(T(D_1)\oplus T(D_2))U_1^{-1}=T(D_3),$$

(3) $$U_2(T(D_1)\otimes T(D_2))U_2^{-1}=T(D_4),$$

for arbitrary unitary equivalence relation U_1(resp. U_2) between $D_1\oplus D_2$(resp. $D_1\otimes D_2$) and D_3(resp. D_4).

In the previous paper [1], we showed,

Proposition. *For any admissible operator field $\boldsymbol{T}$, there exists unique element g in G such that*

$T(D)=U_g^D,$ *for any D in Ω.*

The present work is devoted to prove,

Theorem. *The assumption* (1) *about unitarity of $T(D)$ is replaceable by weaker assumption,*

(1′) *For regular representation R of G, $T(R)$ is a non-zero bounded operator in $L^2(G)$, and $T(D)$ is a closed operator in $\mathfrak{H}^D$ for any D in Ω.*

2. Proof of the theorem.

Lemma. *Under the assumption* (1′),

$$\|T(R)\|=1.$$

In fact, the general theory shows,

$$\|T_1\otimes T_2\|\leqq\|T_1\|\,\|T_2\|,$$

$$\Big\|\sum_\alpha\oplus T_\alpha\Big\|=\sup_\alpha\|T_\alpha\|.$$

While as shown in [1], $R\otimes R$ is equivalent to a multiple of R, so the conditions (2) and (3) lead us to

$$\|T(R)\|^2\geqq\|T(R)\otimes T(R)\|=\|T(R)\|,$$

then $\|T(R)\|\geqq 1$, because of $T(R)\neq 0$. If $\|T(R)\|=a>1$, there exist $\varepsilon>0$ such that $(a-\varepsilon)^2>a$, and a non-zero vector f in $L^2(G)$ such as $\|T(R)f\|>(a-\varepsilon)\|f\|$.

$$\|T(R)\|\,\|f\|^2=\|T(R)\otimes T(R)\|\,\|f\otimes f\|\geqq\|T(R)f\otimes T(R)f\|=$$
$$=\|T(R)f\|^2>(a-\varepsilon)^2\|f\|^2>a\|f\|^2.$$

That contradicts. q.e.d.

The same argument as in [1] concludes $T(R)\,(=T)$ rises a set transformation of G_δ-compact set E to a measurable set $T(E)$ in G, and this map satisfies $\mu(T(E))\leqq\mu(E)$, $T(h)T(f)=T(hf)$, $TL_g=L_gT$, etc.

From the linearity of T and the relation $T(\chi_E)=\chi_{T(E)}$, it is easy to see

$$T(f)\geqq 0, \qquad \text{for} \quad f\geqq 0 \text{ in } L^2(G).$$

Now we consider a function h_E for $h\in C_0^+(G)$ and G_δ-compact set E, defined by

$$h_E(g)=\int h(g_1^{-1})\chi_E(g_1g)d\mu(g_1).$$

Then

$$\int(1/\mu(E))(Th_E)(g)d\mu(g)=(\mu(T(E))/\mu(E))\int h(g)d\mu(g).$$

If E tends to the set $\{e\}$, then the left hand side converges to $\int(Th)(g)d\mu(g)<+\infty$. This assures the existence of

$$\lim_{E\to\{e\}}(\mu(T(E))/\mu(E))=c.$$

Put $h=|h_1|^2$ for given h_1 in $C_0(G)$, we get

$$\|T(h_1)\|^2=c\|h_1\|^2.$$

Since $C_0(G)$ is dense in $L^2(G)$ and $\|T\|=1$, c must be 1 and T is isometric. The proof of proposition 1 in [1] used the fact that T is not unitary but isometric, so the same conclusion is valid in this case too.

It is easy to show, the proof of lemma 2 in [1] is extendable to this case, and combining the result of above discussion, we obtain the proof of the theorem.

Reference

[1] N. Tatsuuma: A duality theorem for locally compact groups. I. Proc. Japan Acad., **41**, 878-882 (1965).