10. Notes on Commutative Archimedean Semigroups. I

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1. Introduction. A commutative semigroup S is archimedean if and only if for any ordered pair of elements, (a, b), of S there are an element c of S and a positive integer n (depending on (a, b)) such that $a^n = bc$. The author proved the following theorem in [5] (or p. 136, [1]).

Theorem 1. *If S is a commutative cancellative archimedean semigroup without idempotent, then S is isomorphic onto the semigroup of all pairs of non-negative integers and elements of an abelian group G,*

$$\{(n, \alpha); n \in N, \alpha \in G\}, \ N=\{0, 1, 2, \cdots\}$$

with a non-negative integer valued function $I: G \times G \to N$ where the multiplication is defined by

$$(n, \alpha)(m, \beta)=(n+m+I(\alpha, \beta), \alpha\beta)$$

and I satisfies

(1.1) $I(\alpha, \beta)=I(\beta, \alpha)$

(1.2) $I(\alpha, \beta)+I(\alpha\beta, \gamma)=I(\alpha, \beta\gamma)+I(\beta, \gamma)$

(1.3) *For any $\xi \in G$, there is a positive integer m depending on ξ such that $I(\xi^m, \xi)>0$.*

(1.4) *$I(\varepsilon, \varepsilon)=1$, ε being the identity of G.*

Further S is homomorphic onto G, $S=\bigcup_{\alpha\in G} S_\alpha$ where each congruence class S_α is a linearly ordered set with respect to the ordering $x<y$ defined by $x=a^n y$ for some positive integer n for a fixed element a of S.

No satisfactory construction theory has been established in the following cases: commutative archimedean semigroups with zero and commutative archimedean semigroups without idempotent in which cancellation is not assumed, except special cases (see [4], [5], [7]).

This paper and the continuation reports the theory of construction of the two cases just mentioned without proof. These results would complete the construction theory of commutative archimedean semigroups in all cases. The detailed paper will be published elsewhere [6].

2. Group decomposition. In § 2 through § 4 S is assumed to be a commutative archimedean semigroup either without idempotent or with zero.

Lemma 1. *If $x \neq 0$, then $x \neq xy$ for all $y \in S$ (see [4], [5]).*

Let a be a fixed non-zero element of S. We define two relations τ and ρ on S:

$x\tau y$ if and only if $x = a^n y$ for some non-negative integer n. "$x\tau y$" will be also denoted by "$x \underset{\tau}{\leq} y$" or "$x \leq y$". If $x \underset{\tau}{\leq}$ but $x \neq y$ then we denote it by $x < y$.

$x\rho y$ if and only if $a^n x = a^m y$ for some non-negative integers n, m. Then τ is a compatible partial ordering, namely S is a naturally partially ordered semigroup with respect to τ. The relation ρ is a congruence on S generated by τ.

Theorem 2. *The factor semigroup S/ρ is a group. If S has a zero, then $\rho = S \times S$ namely S/ρ is a trivial group.*

Thus we have a decomposition of S modulo ρ

$$S = \bigcup_{\lambda \in G} S_\lambda, \quad G \cong S/\rho.$$

Each congruence class S_λ modulo ρ is still a partially ordered set and no element of S_λ is τ-related to any element of S_μ, $\lambda \neq \mu$.

Lemma 3. *For $x \in S$, $ax < x$ and there is no $z \in S$ such that $ax < z < x$. If $b < c$, then there are a finite number of elements x between b and c; the set of x forms a chain.*

Lemma 4. *$x < y$ and $z < y$ imply $x < z$ or $x = z$ or $x > z$.*

Lemma 5. *The ascending chain condition holds:*

$$x_1 < x_2 < \cdots$$

terminates at a finite number of terms.

Thus S has maximal elements with respect to τ. These are called primes (to a). The element a itself is a prime (to a).

Lemma 6. *Each S_λ is a semilattice, namely, any two elements of S_λ have a greatest lower bound in S_λ.*

Thus S is the set union of disjoint semilattices S_λ in the above sense.

3. Trees. Apart from S, we give a concept of trees in general. Let L be a semilattice with respect to $\cap$ in which $a \leq b$ means $a \cap b = a$. The following conditions are equivalent:

(2.1) For any $b < c$, the set $\{x; b \leq x \leq c\}$ is a chain.

(2.2) For any a, the set $\{x; x \leq a\}$ is a chain.

(2.3) For any a, x, $y \in L$ either $a \cap x \geq a \cap y$ or $a \cap x \leq a \cap y$.

(2.4) $x < z$, $y < z$ imply either $x \leq y$ or $x \geq y$.

(2.5) At least two of $a \cap b$, $b \cap c$, and $c \cap a$, are equal for any a, b, $c \in L$.

A semilattice L is called a tree if one of the above conditions holds. A tree is called dispersed if any set of the form $\{x; b \leq x \leq c\}$ is a finite chain.

Let P be a set, N be the set of all non-negative integers.

Consider a mapping of $P\times P$ into $N\times N$

$$(p, q)\rightarrow(h_p(p, q), h_q(p, q))$$

such that $h_p(p, q)$ and $h_q(p, q)$ satisfy the following conditions:

(3.1) $h_p(p, q)\geq 0$ and $h_p(p, q)=0$ if and only if $p=q$

(3.2) $h_p(p, q)=h_p(q, p)$.

(3.3) For any p, q, r, at least one of the following three systems holds

(3.3.1) $h_p(r, p)+h_q(p, q)=h_p(p, q)+h_q(q, r)$,
$h_q(q, r)\geq h_q(q, p)$, $h_r(r, p)=h_r(q, r)$

(3.3.2) $h_q(p, q)+h_r(q, r)=h_q(q, r)+h_r(r, p)$,
$h_r(r, p)\geq h_r(r, q)$, $h_p(p, q)=h_p(r, p)$

(3.3.3) $h_r(q, r)+h_p(r, p)=h_r(r, p)+h_p(p, q)$,
$h_p(p, q)\geq h_p(p, r)$, $h_q(q, r)=h_q(p, q)$.

Let L' be the product set of N and P

$$L'=\{(n, p);\ n\in N,\ p\in P\}.$$

Define a quasi-ordering $\leq$ on L' as follows

(4) $(m, q)\leq(n, p)$ if and only if
$$m-h_q(p, q)\geq 0,\quad n-h_p(p, q)\leq m-h_q(p, q)$$

and then define an equivalence ξ by

(5) $(m, q)\xi(n, p)$ if and only if $(m, q)\leq(n, p)$ and $(m, q)\geq(n, p)$.

Then $L=L'/\xi$ is a partially ordered set respecting the partial ordering induced by $\leq$. A partially ordered set is said to satisfy the abovebounded condition if for any element x there is a maximal element b such that $x\leq b$.

According to [3], we have

Theorem 3. *L is a dispersed tree, without smallest element, satisfying the above bounded condition. Any tree of this kind is isomorphic with the above-mentioned L.*

For the case where a smallest element is, let $w: P\rightarrow N\backslash 0$ be a positive integer valued function defined on P. Consider a mapping of $P\times P$ into $N\times N$, $(p, q)\rightarrow(h_p(p, q),\ h_q(p, q))$ such that $h_p(p, q)$ and $h_q(p, q)$ satisfy (3.1) through (3.3) additionally (3.4) below

$$(3.4)\qquad \begin{cases} h_p(p, q)\leq w(p) \text{ for all } q\in P \\ h_q(p, q)\leq w(q) \text{ for all } p\in P. \end{cases}$$

L' is defined as the same as the previous case and ξ_1 is defined as follows:

$$\xi_1=\xi\cup\xi'$$

where ξ is defined by (5), and ξ' is defined as follows:

$(m, q)\xi'(n, p)$ for all m, n, p, q such that $n\geq w(p)$, $m\geq w(q)$.

Let $L_1=L'/\xi_1$. Then L_1 is a partially ordered set with respect to the partial ordering induced by $\leq$.

Theorem 4. *L_1 is a dispersed tree, with smallest element,* *satisfying the above-bounded condition. Any tree of this kind is isomorphic onto L_1.*

4. The Congruence classes. Returning to archimedean semigroup S, we can say in the following way:

If S has no idempotent then each S_λ is a dispersed tree, without smallest element, satisfying the ascending chain condition; if S has a zero, then S is a dispersed tree, with smallest element, satisfying the ascending chain condition.

The functions $h_p(p, q)$, $h_q(p, q)$ given in S_λ are denoted by $h_p^{(\lambda)}(p, q)$, $h_q^{(\lambda)}(p, q)$, respectively.

We define a relation η on S as follows:

$x\eta y$ if and only if $a^n x = a^n y$ for some positive integer n where a is originally fixed.

Theorem 5. *The relation η is the smallest cancellative congruence on S. If S has no idempotent, S/η is a commutative archimedean semigroup without idempotent. If S has a zero, $\eta = S\times S$.*

By a maximal ascending chain from b we mean a sequence of elements of S

$$b = b_0 < b_1 < b_2 < \cdots < b_n = p$$

where p is a prime and there is no element between b_{i-1} and $b_i (i = 1, \cdots, n)$. The number n is called the length of the maximal ascending chain from b.

Lemma 7. *For a fixed $b \in S$, the set of the lengths of all maximal chains from b is bounded.*

By a maximal descending chain from a prime p we mean a sequence of elements of S

$$\boldsymbol{C}: p = c_0 > c_1 > c_2 > \cdots > c_n > \cdots$$

where $c_{n-1} > x > c_n$ for no $x \in S$. For a fixed $\boldsymbol{C}$, let $f(n)$ denote the maximum of the lengths of all maximal ascending chains from c_n. Let $f(0) = 0$.

Lemma 8. *The set $\{f(n) - n;\ n = 0, 1, 2, \cdots\}$ is bounded above, namely, there is a non-negative integer k such that*

$$f(n) - n \leq k \text{ for all } n.$$

This lemma can be restated as follows: Let p be a fixed prime in S_λ. There is a prime p_0 in S_λ such that

$$h_{p_0}^{(\lambda)}(p_0, p) - h_p^{(\lambda)}(p_0, p) \geq h_{p_\alpha}^{(\lambda)}(p_\alpha, p) - h_p^{(\lambda)}(p_\alpha, p) \tag{6}$$

for all primes p_α in S_λ.

For any prime p_α in S_λ we define non-negative integer-valued functions as follows:

$$\begin{cases} \sigma_\lambda(p_\alpha) = h_{p_0}^{(\lambda)}(p_0, p_\alpha) - h_{p_\alpha}^{(\lambda)}(p_0, p_\alpha), \\ \pi_\lambda(p_\alpha) = h_{p_0}^{(\lambda)}(p_0, p_\alpha). \end{cases} \tag{7}$$

Lemma 9. *A prime p_0 satisfies (6) if and only if* $\sigma_\lambda(p_\alpha)\geq 0$ *for all primes* p_α *in* S_λ.

Such a prime p_0 is called a highest prime.

Theorem 6. *If S has no idempotent, then each S_λ, $\lambda\in G$, is a dispersed tree without smallest element. If S has a zero, then $|G|=1$, S is a dispersed tree with smallest element. In both cases the trees satisfy the ascending chain condition, and each S_λ has at least one highest prime to a.*

5. Construction of special case. Assume that the following systems and functions are given:

(8.1) G: an abelian group whose elements are denoted by λ, $\mu,\cdots$

(8.2) N: the set of all non-negative integers

(8.3) $I(\lambda, \mu)$: a function of $G\times G$ into N satisfying (1.1), (1.2), (1.3), and (1.4) in Theorem 1.

(8.4) $\{P_\lambda; \lambda\in G\}$: a family of disjoint sets P_λ with functions $h_\alpha^{(\lambda)}(\alpha, \beta)$, $h_\beta^{(\lambda)}(\alpha, \beta)$ satisfying (3.1) through (3.3) such that
$$\sigma_\lambda(\alpha)\geq 0 \text{ and } \pi_\lambda(\alpha)\leq\sigma_\lambda(\alpha)+1 \text{ for all } \alpha\in P_\lambda.$$

A chosen highest prime in S_λ is denoted by ι_λ. (S_λ denotes L in § 2.)

Let ξ_λ denote the relation (5) on S_λ, and ζ_λ the relation (4) $\leq$ on S_λ and let
$$\xi=\bigcup_{\lambda\in G}\xi_\lambda,\ \zeta=\bigcup_{\lambda\in G}\zeta_\lambda,\ \xi\mid S_\lambda=\xi_\lambda,\ \zeta\mid S_\lambda=\zeta_\lambda$$
where $\xi\mid S_\lambda$ is the restriction of ξ to S_λ.

Then the set $(N\times P)/\xi$ is a partially ordered set and
$$(N\times P)/\xi=\bigcup_{\lambda\in G}((N\times P_\lambda)/\xi_\lambda).$$
Let $h_\alpha(\alpha, \beta)$ be the function on $P\times P$ obtained by uniting $h_\alpha^{(\lambda)}(\alpha, \beta)$ through $\lambda\in G$:
$$h_\alpha(\alpha, \beta)=\bigcup_{\lambda\in G}h_\alpha^{(\lambda)}(\alpha, \beta),\ h_\beta(\alpha, \beta)=\bigcup_{\lambda\in G}h_\beta^{(\lambda)}(\alpha, \beta)$$
and also
$$\sigma(\alpha)=\bigcup_{\lambda\in G}\sigma_\lambda(\alpha),\ \pi(\alpha)=\bigcup_{\lambda\in G}\pi_\lambda(\alpha).$$
For convenience we state the definition of ξ:

(9) $(n, \alpha)\xi(m, \beta)$ if and only if α and β are in a same P_λ for some $\lambda\in G$, and $n-h_\alpha(\alpha, \beta)=m-h_\beta(\alpha, \beta)\geq 0$.

We define a binary operation on P as follows:

For any $\alpha_\lambda\in P_\lambda$ and $\beta_\mu\in P_\mu$, choose a prime $\gamma_{\lambda\mu}\in P_{\lambda\mu}$ such that

(10) $\sigma(\alpha_\lambda)+\sigma(\beta_\mu)+I(\lambda, \mu)\geq\pi(\gamma_{\lambda\mu})$.

This is always possible since we can take at least $\gamma_{\lambda\mu}=\iota_{\lambda\mu}$. Such $\gamma_{\lambda\mu}$ is not unique but we choose one of those, and $\gamma_{\lambda\mu}$ is denoted by
$$\gamma_{\lambda\mu}=\alpha_\lambda\beta_\mu.$$
Additionally we can require
$$(11)\qquad \begin{cases}\alpha_\lambda\beta_\mu=\beta_\mu\alpha_\lambda & \text{for all } \alpha_\lambda,\ \beta_\mu\\ \alpha_\lambda\iota_\varepsilon=\iota_\varepsilon\alpha_\lambda=\alpha_\lambda & \text{for all } \alpha_\lambda.\ (\varepsilon \text{ is the identity of } G)\end{cases}$$

Thus we have a commutative groupoid P with identity ι_ε, and the groupoid P is homomorphic onto G.

Next we define a function $K(\alpha_\lambda, \beta_\mu)$ on $P \times P$ as follows:

(12) $$K(\alpha_\lambda, \beta_\mu) = \sigma(\alpha_\lambda) + \sigma(\beta_\mu) + I(\lambda, \mu) - \sigma(\alpha_\lambda \beta_\mu)$$

where $\alpha_\lambda \beta_\mu$ is the product in the groupoid P just obtained. K is a non-negative integer valued function and immediately

(13) $$\begin{cases} K(\alpha_\lambda, \beta_\mu) = K(\beta_\mu, \alpha_\lambda) \\ K(\alpha_\lambda, \iota_\varepsilon) = 1 \text{ for all } \alpha_\lambda, \text{ all } \lambda \in G. \end{cases}$$

For the given groupoid P, and for the given function K, we define a binary operation on S:

(14) $$(n, \alpha_\lambda)(m, \beta_\mu) = (n + m + K(\alpha_\lambda, \beta_\mu), \alpha_\lambda \beta_\mu).$$

This definition (14) is given for elements of $N \times P$, but we can show that it is single valued on $S = (N \times P)/\xi$ and that the system S with (14) satisfies all our requirements.

Theorem 7. *The S constructed above is a commutative archimedean semigroup without idempotent.*

Theorem 7 gives the construction of a special case. The general case together with commutative archimedean semigroups with zero will be discussed in the second part of this paper.

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