3. *A Remark on a Periodic Boundary Problem of Parabolic Type*

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Let $Q=\{(t,x):\ -\infty<t<+\infty,\ x=(x_1,\cdots,x_n)\in\Omega\}$ and $\partial Q=\{(t,x):\ -\infty<t<+\infty,\ x\in\partial\Omega\}$, where Ω is a bounded domain in euclidean n-space E^n with boundary $\partial\Omega$. Let there be given the system of semilinear parabolic equations

(1) $\mathcal{L}_i(u_i)=f_i(t,x,u_1,\cdots,u_N)$ in Q $(i=1,\cdots,N)$

and the boundary condition

(2) $u_i=\varphi_i(t,x)$ on ∂Q $(i=1,\cdots,N)$,

where

$$\mathcal{L}_i=\frac{\partial}{\partial t}-\sum_{p,q=1}^{n}\frac{\partial}{\partial x_p}a_{pq}^i(t,x)\frac{\partial}{\partial x_q}+\sum_{p=1}^{n}b_p^i(t,x)\frac{\partial}{\partial x_p}$$

and the functions a_{pq}^i, b_p^i, f_i, and φ_i $(p,q=1,\cdots,n;\ i=1,\cdots,N)$ are periodic in t with period $T(T>0)$. In the present note we shall be concerned with the problem of finding a solution, periodic in t with period T, to the boundary problem (1), (2) which will be called the first periodic boundary problem of parabolic type.

We introduce the following assumptions:*)

I. There is a positive constant λ snch that, for any real vector ξ and for all $(t,x)\in\bar{Q}$,

$$\sum_{p,q=1}^{n}a_{pq}^i(t,x)\xi_p\xi_q\geqq\lambda\sum_{p=1}^{n}\xi_p^2\qquad(i=1,\cdots,N).$$

II. $a_{pq}^i\in C^{1+\alpha}(\bar{Q})$ and $b_p^i\in C^{\alpha}(\bar{Q})$ $(0<\alpha<1)$ $(p,q=1,\cdots,n;\ i=1,\cdots,N)$.

III. The functions $f_i(t,x,z_1,\cdots,z_N)$ $(i=1,\cdots,N)$ are defined in $\mathcal{D}=\{(t,x,z_1,\cdots,z_N):\ (t,x)\in\bar{Q},\ -\infty<z_k<+\infty,\ k=1,\cdots,N\}$, are in $C^{\alpha}(\bar{Q})$ for each fixed $(z_1,\cdots,z_N)$, and satisfy the Lipschitz condition $|f_i(t,x,z_1,\cdots,z_N)-f_i(t,x,\bar{z}_1,\cdots,\bar{z}_N)|\leqq l_i\sum_{p=1}^{n}|z_p-\bar{z}_p|$ $(i=1,\cdots,N)$. Moreover, the system $\{f_i\}$ is quasi-monotone increasing in $z_1,\cdots,z_N$; that is, for each i and for $z_k\leqq\bar{z}_k$ $(k=1,\cdots,N)$, $z_i=\bar{z}_i$, the inequality

$$f_i(t,x,z_1,\cdots,z_N)\leqq f_i(t,x,\bar{z}_1,\cdots,\bar{z}_N)$$

holds.

IV. $\Omega\in A^{2+\alpha}$; $\varphi_i\in C^{2+\alpha}(\partial Q)$.

V. There exist functions $\underline{\omega}_i(t,x)$, $\overline{\omega}_i(t,x)$ $(\underline{\omega}_i\leqq\overline{\omega}_i)$ in $C^{\alpha}(\bar{Q})$

*) For the definitions of $C^{r+\alpha}(\overline{Q})$ $(r=0,1,2)$, $C^{2+\alpha}(\partial Q)$, and $A^{2+\alpha}$ see [2], [6].

which are periodic in t with period T and satisfy the following inequalities:

$$
(3)\qquad \begin{aligned}
&\mathcal{L}_i(\underline{\omega}_i)\leqq f_i(t, x, \underline{\omega}_1, \cdots, \underline{\omega}_N) \quad \text{in} \quad Q,\\
&\mathcal{L}_i(\overline{\omega}_i)\geqq f_i(t, x, \overline{\omega}_1, \cdots, \overline{\omega}_N) \quad \text{in} \quad Q,\\
&\underline{\omega}_i(t, x)\leqq \varphi_i(t, x)\leqq \overline{\omega}_i(t, x) \quad \text{on} \quad \partial Q \qquad (i=1, \cdots, N).
\end{aligned}
$$

Theorem. *Under Assumptions* I–V *there exists a solution, periodic in* t *with period* T, *to the periodic boundary problem* (1), (2).

Proof. We proceed by arguments used in our previous papers [3, 4]. (See also Brzychczy [1]). Let us construct the sequence of systems of functions $\{v_i^{(m)}(t, x)\}$ $(i=1, \cdots, N)$, $m = 0, 1, 2, \cdots$, by determining successively the solutions, periodic in t with period T, of the following linear parabolic equations

$$
(4)\qquad \Lambda_i(v_i^{(m)})\equiv \mathcal{L}_i(v_i^{(m)})+l_i v_i^{(m)}=f_i^{(m-1)}(t, x) \quad \text{in} \quad Q
$$

satisfying the boundary condition (2), where we have set

$$
\begin{aligned}
&f_i^{(m-1)}(t, x)\equiv f_i(t, x, v_1^{(m-1)}, \cdots, v_N^{(m-1)})+l_i v_i^{(m-1)}\\
&v_i^{(0)}(t, x)\equiv \overline{\omega}_i(t, x) \qquad (i=1, \cdots, N;\ m=1, 2, \cdots).
\end{aligned}
$$

The existence and the uniqueness of such solutions $v_i^{(m)}$ follow from the theorems of Fife [2] and Shmulev [6].

It can be proved that the inequalities

$$
(5)\qquad \underline{\omega}_i(t, x)\leqq v_i^{(m)}(t, x)\leqq v_i^{(m-1)}(t, x)\leqq \overline{\omega}_i(t, x) \qquad (i=1, \cdots, N)
$$

hold in $\bar{Q}$ for all $m=1, 2, \cdots$. Indeed, noting that

$$
\Lambda_i(\overline{\omega}_i-v_i^{(1)})\geqq 0 \text{ in } Q \text{ and } \overline{\omega}_i-v_i^{(1)}\geqq 0 \text{ on } \partial Q \qquad (i=1, \cdots, N)
$$

and applying the maximum principle [4], we get

$$
v_i^{(1)}(t, x)\leqq \overline{\omega}_i(t, x) \quad \text{in} \quad \bar{Q} \qquad (i=1, \cdots, N).
$$

We observe that

$$
(6)\qquad \begin{aligned}
\Lambda_i(v_i^{(1)}-\underline{\omega}_i)\geqq &[f_i(t, x, \overline{\omega}_1, \cdots, \underline{\omega}_i, \cdots, \overline{\omega}_N)-f_i(t, x, \underline{\omega}_1, \cdots, \underline{\omega}_N)]\\
&+[f_i(t, x, \overline{\omega}_1, \cdots, \overline{\omega}_N)-f_i(t, x, \overline{\omega}_1, \cdots, \underline{\omega}_i, \cdots, \overline{\omega}_N)\\
&\qquad +l_i(\overline{\omega}_i-\underline{\omega}_i)] \quad \text{in} \quad Q,\\
v_i^{(1)}-\underline{\omega}_i\geqq 0 &\quad \text{on} \quad \partial Q \qquad (i=1, \cdots, N).
\end{aligned}
$$

Since the right-hand side of (6) is non-negative by virtue of the quasi-monotony and the Lipschitz continuity of f_i and in view of (3), it follows from the maximum principle that

$$
\underline{\omega}_i(t, x)\leqq v_i^{(1)}(t, x) \quad \text{in} \quad \bar{Q} \qquad (i=1, \cdots, N).
$$

The proof of (5) for general m is similar. Thus, the sequence $\{v_i^{(m)}\}$ is uniformly bounded and monotone non-increasing for each i.

Applying the theorem of Ladyzhenskaia-Ural'tseva [5] to (4), (2), we find that $v_i^{(m)}\in C^{\beta}(\bar{Q})$ for some $\beta(0<\beta<1)$ and that

$$
|v_i^{(m)}|_{\beta}\leqq \text{constant independent of } m \qquad (i=1, \cdots, N).
$$

Consequently, by the theorem of Shmulev [6], we obtain

$$
|v_i^{(m)}|_{2+\alpha\beta}\leqq \text{const } (|f_i^{(m-1)}|_{\alpha\beta}+|\varphi_i|_{2+\alpha}) \qquad (i=1, \cdots, N),
$$

the right-hand side of which is bounded by a constant independent of m. It is now easy to conclude that the limit functions

$$v_i(t, x) = \lim_{m\to\infty} v_i^{(m)}(t, x) \qquad (i=1, \cdots, N)$$

constitute the periodic solution sought to the boundary problem (1), (2). The proof is thus completed.

Remark 1. If all the a^i_{pq}, b^i_p, f_i, and φ_i are time-independent, then so is the solution. Therefore, the elliptic boundary problem

$$-\sum_{p,q=1}^{n} \frac{\partial}{\partial x_p} a^i_{pq}(x) \frac{\partial u_i}{\partial x_q} + \sum_{p=1}^{n} b^i_p(x) \frac{\partial u_i}{\partial x_p} = f_i(x, u_1, \cdots, u_N) \quad \text{in} \quad \Omega,$$

$$u_i = \varphi_i(x) \quad \text{on} \quad \partial\Omega \qquad (i=1, \cdots, N)$$

may be regarded as a special case of the periodic boundary problem of parabolic type (1), (2).

Remark 2. i) If $z_i f_i(t, x, z_1, \cdots, z_N) \leqq -a_i^0 z_i^2 + a_i^1$, $a_i^0 > 0$ and $a_i^1 \geqq 0$ being constants $(i=1, \cdots, N)$, then the functions $\underline{\omega}_i$ and $\overline{\omega}_i$ defined by

$$\overline{\omega}_i(t, x) \equiv -\underline{\omega}_i(t, x) \equiv \text{const} > \max\{\sup_{\partial Q} |\varphi_i|, \sqrt{a_i^1/a_i^0}\} \quad (i=1, \cdots, N)$$

satisfy Assumption V.

ii) If f_i are bounded in $\mathcal{D}(|f_i| \leqq M_i)$, then the periodic solution (period T) to the boundary problem

$\mathcal{L}_i(u_i) = M_i[-M_i]$ in Q and $u_i = \Phi_i[-\Phi_i]$ on ∂Q $(i=1, \cdots, N)$, where $\Phi_i = \text{const} > \sup_{\partial Q} |\varphi_i|$, plays the part of $\{\overline{\omega}_i(t, x)\}[\{\underline{\omega}_i(t, x)\}]$ in Assumption V.

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