59. *Fundamental Equations of Branching Markov Processes*

By Nobuyuki IKEDA, Masao NAGASAWA, and Shinzo WATANABE
Osaka University, Tokyo Institute of Technology, and Kyoto University

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We have given in the previous paper [2] a definition of branching Markov processes and discussed some fundamental properties of them. Here we shall treat several fundamental equations which describe and characterize these processes.

1. Fundamental quantities of branching Markov processes. In this paper we shall use constantly the notation[1] and the terminolog adopted in [2].

Definition 1.1. Let X_t be a branching Markov process (abbreviated as B.M.P.) on $\boldsymbol{S}$. We denote the killed process on S^n of X_t at the first branching time τ by $X_t^{(0)}$ and call it the *non branching part* on S^n of B.M.P. X_t. The non branching part on S^1 is called simply the non branching part of X_t, and its semi-group on $\boldsymbol{B}(S)$ is defined by

$$(1.1)\qquad T_t^0 f(x)=E_x[f(X_t);\, t<\tau],\qquad f\in \boldsymbol{B}(S),\ x\in S.$$

Further we denote

$$(1.2)\qquad K(x, dt, dy)=P_x[\tau\in dt,\, X_{\tau-}\in dy],\qquad x\in S,\ dy\subset S.$$ [2]

Definition 1.2. Assume that there exists a system $\{q_n(x);\, n=0, 2, 3, \cdots, +\infty\}$ of non-negatives Borel measurable functions on S and a system $\{\pi_n(x, d\boldsymbol{y});\, n=0, 2, \cdots, +\infty\}$ of non-negatives kernels[3] on $S\times \boldsymbol{S}$ such that

$$(1.3)\qquad P_x[X_\tau\in d\boldsymbol{y}\,|\,X_{\tau-}]=\pi(X_{\tau-}, d\boldsymbol{y}),$$

almost surely (P_x) on $\{\tau<\infty\}$, $x\in S$, $d\boldsymbol{y}\subset \boldsymbol{S}$, where we put

$$(1.4)\qquad \pi(x, d\boldsymbol{y})=\sum_{n=0}^{\infty}{}^{*}\, q_n(x)\pi_n(X, d\boldsymbol{y}\cap S^n),$$

and $\sum_{n=0}^{\infty}{}^{*}$ denotes the sum over $n=0, 2, \cdots, +\infty$ and $S^\infty=\{\Delta\}$. Then we shall call $\{q_n, \pi_n, n=0, 2, \cdots, +\infty\}$ the *branching system* of B.M.P. X_t. It is clear that if a kernel $\pi(x, d\boldsymbol{y})$ on $S\times\boldsymbol{S}$ satisfing (1.3) is given, then the system

$$(1.5)\qquad q_n(x)=\pi(x, S^n),\ \pi_n(x, d\boldsymbol{y})=\pi(x, d\boldsymbol{y})/q_n(x),\qquad n=0, 2, \cdots, +\infty,$$

is the branching system of B.M.P. X_t.

The above defined $\{T_t^0, K, q_n, \pi_n\}$ are fundamental quantities of B.M.P. which completely determine the B.M.P. X_t. In this paper

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1) In [2], branching Markov processes are denoted by x_t, but in the following we write it as X_t.

2) We write as $X_{\sigma-}=\lim_{t\uparrow\sigma} X_t$, for any random time σ.

3) $\pi(x, d\boldsymbol{y})$ is said to be a non-negative kernel on $S\times\boldsymbol{S}$, if for any Borel set $B\subset\boldsymbol{S}$, $\pi(., B)$ is a Borel measurable function on S and for any $x\in S$, $\pi(x, .)$ is a non-negative measure on $\boldsymbol{S}$ with total mass less than 1.

we shall discuss some equations defined through $\{T_t^0, K, q_n, \pi_n\}$. More detailed study on these equations and the construction of a B.M.P. through these equations will be given in the forthcoming papers.

Now, we assume in the following $\{T_t^0, K, q_n, \pi_n\}$ are given *a prioi* independent of B.M.P. X_t. Namely,

Definition 1.3. Let $\{T_t^0, K, q_n, \pi_n\}$ be a system satisfying the following conditions:

1°) Given a Markov process $X^0=\{X_t^0, \zeta^0, P_x^0, x\in S\}$ on S and assume $X^0_{\zeta^0-}\in S$ exists, where ζ^0 is the life time of X^0, then T_t^0 and K are defined by

$$(1.6)\qquad \begin{cases} T_t^0 f(x)=E_x^0[f(X_t^0);\, t<\zeta^0], & x\in S, f\in \boldsymbol{B}(S), \text{ and} \\ K(x, dt, dy)=P_x^0[\zeta^0\in dt, X^0_{\zeta^0-}\in dy], & x\in S, dy\subset S. \end{cases}$$

2°) a) $\{q_n(x); n=0, 2, \cdots, +\infty\}$ is a system of non-negative measurable functions such as

$$(1.7)\qquad \sum_{n=0}^{\infty}{}^{*}\, q_n(x)=1,\ x\in S, \quad \text{and}$$

b) $\{\pi_n(x, dy); n=0, 2, \cdots, +\infty\}$ is a system of non-negative kernels on $S\times S^n$ satisfying

$$(1.8)\qquad \pi_n(x, S^n)=1,\ x\in S,\ n=0, 2, \cdots, +\infty.$$

Then we call $\{T_t^0, K, q_n, \pi_n\}$ (or $\{T_t^0, K, \pi\}$ where π is given by (1.5)) a *fundamental system*. If these are given by Definitions 1.1 and 1.2., we call them the *fundamental system of B.M.P.* X_t.

2. Some preparatory results. We need the following Lemmas. Put

$$(2.1)\qquad \begin{aligned} &\boldsymbol{C}^*(S)=\{f; f\in \boldsymbol{C}(S) \text{ and } \|f\|<1\}, \quad \text{and} \\ &\overline{\boldsymbol{C}}^*(S)=\{f; f\in \boldsymbol{C}(S) \text{ and } \|f\|\leqq 1\}. \end{aligned}$$

The non-negative part of $\boldsymbol{C}^*(S)$ is denoted by $\boldsymbol{C}^*(S)^+$.

Lemma 2.1. i) The linear hull of $\{\hat{f}|_{S^n}; f\in \boldsymbol{C}^*(S)^+\}$ is dense in $\boldsymbol{C}(S^n)$. ii) The linear hull of $\{\hat{f}; f\in \boldsymbol{C}^*(S)^+\}$ is dense in $\boldsymbol{C}_0(\boldsymbol{S})$.[4]

Lemma 2.2. Let $\nu_1, \nu_2, \cdots, \nu_k$ be signed measures on $\boldsymbol{S}-\{\varDelta\}$ of bounded total variations. Then i) there exists one and only one signed measure μ on $\boldsymbol{S}-\{\varDelta\}$, such as

$$\int_{\boldsymbol{S}-\{\varDelta\}} \hat{f}\,d\mu=\prod_{j=1}^{k}\int_{\boldsymbol{S}-\{\varDelta\}} \hat{f}\,d\nu_j, \qquad \text{for any } f\in \boldsymbol{C}^*(S).$$

We denote

$$\mu=\nu_1\otimes\nu_2\otimes\cdots\otimes\nu_k.$$

Then we have ii)

$$|\mu|=|\nu_1|\otimes|\nu_2|\otimes\cdots\otimes|\nu_k|^{5)}$$

and

$$\mu(\boldsymbol{S}-\{\varDelta\})=\prod_{j=1}^{k}\nu_j(\boldsymbol{S}-\{\varDelta\}).$$

Hence if ν_j are non-negative, μ is non-negative, and if ν_j are probability measures then μ is so.

Definition 2.1. For $f\in \boldsymbol{B}^*(S)$ and $g\in \boldsymbol{B}(S)$[6] put

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4) $\boldsymbol{C}(S)=\{f; f$ is bounded continuous on $\boldsymbol{S}\}$, $\boldsymbol{C}(S^n)=\{f; f$ is bounded continuous on $S^n\}$, and $\boldsymbol{C}_0(\boldsymbol{S})=\{f; f$ is bounded continuous on $\boldsymbol{S}$ with $f(\varDelta)=0\}$.

5) $|\mu|$ and $|\nu_j|$ denote the total variations of μ and ν_j, respectively.

6) $\boldsymbol{B}(S)=\{f; f$ is bounded Borel measurable$\}$, $\boldsymbol{B}^*(S)(\overline{\boldsymbol{B}}^*(S))=\{f; f\in \boldsymbol{B}(S), \|f\|<1$ (resp. $\|f\|\leq 1)\}$.

(2.2) $$\langle f\mid g\rangle(\boldsymbol{x})=\begin{cases}\sum_{k=1}^{n} g(x_k)\prod_{j\neq k} f(x_j), & \text{if } \boldsymbol{x}\in S^n \text{ and } (x_1,\cdots,x_n)\in\boldsymbol{x},\\ 0, & \text{if } \boldsymbol{x}=\partial \text{ or } \Delta.\end{cases}$$

If $f\in \boldsymbol{C}^*(S)$ and $g\in \boldsymbol{C}(S)$, then clearly $\langle f\mid g\rangle\in \boldsymbol{C}_0(\boldsymbol{S})$.

Definition 2.2. For $f\in \overline{\boldsymbol{B}^*}(S)$ put

(2.3) $$F[x,f]=\int_{\boldsymbol{S}}\pi(x,d\boldsymbol{y})\hat{f}(\boldsymbol{y}).$$

Then F defines a *non-linear* operator on $\overline{\boldsymbol{B}^*}(S)$ into $\boldsymbol{B}(S)$.

Theorem 2.1. *Let $\{T_t^0, K, \pi\}$ be a fundamental system of Definition* 1.3. *Then:*

1°) *For $n=0,2,3,\cdots,+\infty$*[7] *there exists a unique non-negative kernel $\boldsymbol{T}^0(t,\boldsymbol{x},d\boldsymbol{y})$ on $S^n\times S^n$ such as*

(2.4) $$\int_{S^n}\boldsymbol{T}^0(t,\boldsymbol{x},d\boldsymbol{y})\hat{f}(\boldsymbol{y})=\widehat{T_t^0 f}(\boldsymbol{x}),\quad \textit{for any } f\in\boldsymbol{C}^*(S) \textit{ and } \boldsymbol{x}\in S^n,$$

and with $\boldsymbol{T}^0(t,\boldsymbol{x},S^n)\leqq 1$.

2°) *There exists a unique non-negative kernel $\Psi(\boldsymbol{x},ds,d\boldsymbol{y})$ on $\boldsymbol{S}\times([0,\infty)\times\boldsymbol{S})$ such as*

(2.5) $$\int_0^t\int_{\boldsymbol{S}}\Psi(\boldsymbol{x},ds,d\boldsymbol{y})\hat{f}(\boldsymbol{y})=\int_0^t\left\langle T_s^0 f\,\middle|\,\int_{\boldsymbol{S}}K(.,ds,dz)F[z,f]\right\rangle(\boldsymbol{x}),\quad \textit{for any } f\in\boldsymbol{C}^*(S),$$

and

(2.6) $$\Psi(\boldsymbol{x},[0,t]\times\boldsymbol{S})=1-\boldsymbol{T}^0(t,\boldsymbol{x},S^n),\ \boldsymbol{x}\in S^n,\ n=0,2,\cdots,+\infty.$$

3°) *For $f\in B(\boldsymbol{S})$, we put*

$$\boldsymbol{T}_t^0 f(\boldsymbol{x})=\int_{\boldsymbol{S}}\boldsymbol{T}^0(t,\boldsymbol{x},d\boldsymbol{y})f(\boldsymbol{y}),\ \boldsymbol{x}\in S^n,\ n=0,2,3,\cdots,+\infty.$$

Then $\{\boldsymbol{T}_t^0,\Psi\}$ satisfies

$$\boldsymbol{T}_t^0 1+\Psi(\boldsymbol{x},[0,t]\times\boldsymbol{S})=1,$$

and

$$\Psi(\boldsymbol{x},[0,t]\times d\boldsymbol{y})=\Psi(\boldsymbol{x},[0,r]\times d\boldsymbol{y})+\boldsymbol{T}_r^0\{\Psi(.,[0,t-r]\times d\boldsymbol{y})\}(\boldsymbol{x}),\ 0\leqq r<t.$$[8]

3. Fundamental equations of B.M.P. In this section we assume that we are given a fundamental system $\{T_t^0, K, \pi\}$ of Definition 1.3., and let $\boldsymbol{T}_t^0$ and Ψ be those of Theorem 2.1.

3.1. M-equation.

Definition 3.1. For $f\in\boldsymbol{C}(\boldsymbol{S})$, consider the following equation

(3.1) $$u_t(\boldsymbol{x})=\boldsymbol{T}_t^0 f(x)+\int_0^t\int_{\boldsymbol{S}}\Psi(\boldsymbol{x},dr,d\boldsymbol{y})u_{t-r}(\boldsymbol{y}),\ \boldsymbol{x}\in\boldsymbol{S},$$

which we call *Moyal equation* (*M-equation*) corresponding to $\{T_t^0, K, \pi\}$. A solution of (3.1) is called a solution of M-equation for the initial value f.

Theorem 3.1. *Suppose that a B.M.P. X_t has the branching system and satisfies the condition* (c. 3) *of Theorem* 1 *in* [2]. *Let $\boldsymbol{T}_t$ be the semi-group of B.M.P. X_t. Then*

$$u_t(\boldsymbol{x})=\boldsymbol{T}_t f(\boldsymbol{x})$$

is a solution of M-equation corresponding to the system $\{T_t^0, K, \pi\}$ of X_t for the initial value $f\in\boldsymbol{C}_0(\boldsymbol{S})$.

Proof is easily performed using Theorem 1 in [2] and so-called Dynkin's formula [1].

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7) For $n=+\infty$, put $T^0(t,\Delta,\{\Delta\})=1$.
8) Moyal [3] called this (P^0,Ψ)-condition.

3.2. S-equation.

Definition 3.2. Consider for $f \in C(S)$, the following equation

$$u_t(x) = T_t^0(x) + \int_0^t \int_S K(x, ds, dy) F[y, u_{t-s}], \; x \in S, \tag{3.2}$$

and we call it *Skorohod equation* (*S-equation*) corresponding to $\{T_t^0, K, \pi\}$.[9]

Theorem 3.2. (*Skorohod* [5]) *Suppose a B.M.P. X_t has the branching system, then*

$$u_t(x) = \boldsymbol{T}_t \hat{f}(x), \; x \in S,$$

is a solution of S-equation corresponding to the system $\{T_t^0, K, \pi\}$ of X_t for initial value $f \in C(S)$, where $\boldsymbol{T}_t$ is the semi-group of B.M.P. X_t.

3.3. Semi-linear parabolic equation (backward equation).

We now set an assumption.

Assumption 1. $\{T_t^0, K, \pi\}$ of Definition 1.3 satisfies the following conditions:

1°) The Markov process X_t^0 in 1°) of Def. 1.3. is obtained as follows: Given a strongly continuous semi-group U_t on $C(S)$ satisfying $U_t 1 = 1$, and a function $k \in C(S)^+$, let $\tilde{X}_t$ be the Hunt process corresponding to U_t. Then X_t^0 is the $\exp\left(-\int_0^t k(\tilde{X}_s) ds\right)$-subprocess of $\tilde{X}_t$.

2°) The kernel $\pi(x, d\boldsymbol{y})$ defines $F[., f] \in C(S)^+$ for any $f \in C^*(S)^+$.[10]

Now let $\mathfrak{G}$ (resp. $\mathfrak{G}^0$) be the generator (Hille-Yosida sense [6]) of U_t (resp. T_t^0) and $\mathfrak{D}(\mathfrak{G})$ (resp. $\mathfrak{D}(\mathfrak{G}^0)$) be its domain, then we have

$$\mathfrak{D}(\mathfrak{G}) = \mathfrak{D}(\mathfrak{G}^0) \quad \text{and} \quad \mathfrak{G}^0 = \mathfrak{G} - k,$$

and K is given by (cf. [4])

$$\int_0^t \int_S K(x, ds, dy) f(y) = \int_0^t T_s^0(kf)(x) ds, \; f \in C(S), \; x \in S.$$

Definition 3.3. Consider the following equation

$$\begin{aligned} \frac{\partial u_t}{\partial t} &= \mathfrak{G}^0 u_t + kF[., u_t], \\ &= \mathfrak{G} u_t + k(F[., u_t] - u_t), \end{aligned} \tag{3.3}$$

and we call it the *semi-linear parabolic equation* (*backward equation*) corresponding to $\{T_t^0, K, \pi\}$.

Theorem 3.3. *Suppose a B.M.P. X_t has the branching system and satisfies the Assumption* 1. *Then* i) *the semi-group $\boldsymbol{T}_t$ of X_t is strongly continuous on $\boldsymbol{C}_0(\boldsymbol{S})$. Let $\boldsymbol{G}$ be its ganerator. If $0 \leqq f < 1$ and $f \in \mathfrak{D}(\mathfrak{G}^0)$, then $\hat{f} \in \mathfrak{D}(\boldsymbol{G})$ and*

$$\boldsymbol{G}\hat{f}(\boldsymbol{x}) = \langle f \mid \mathfrak{G}^0 f + kF[., f] \rangle(\boldsymbol{x}), \; \boldsymbol{x} \in \boldsymbol{S}. \tag{3.4}$$

ii) *For $0 \leqq f < 1$ and $f \in \mathfrak{D}(\mathfrak{G}^0)$,*

$$u_t(x) = \boldsymbol{T}_t \hat{f}(x), \; x \in S,$$

is a solution of (3.3) *corresponding to the system $\{T_t^0, K, \pi\}$ of X_t, which satisfies*

$$\| u_t - f \| \to 0 \;\; (t \downarrow 0).$$

9) Notice that M-equation is an equation on $\boldsymbol{S}$, while S-equation is defined on S.

10) If T_t^0 is strongly Feller, we need not assume 2°).

3.4. Forward equation. Assumption 1 is set. Let

$$\mathfrak{D}^+=\{f; f\in \boldsymbol{C}(S),\ 0<f<1\},$$

and let $A(f)$ be a functional defined on $\mathfrak{D}^+$. A *functional derivative of* $A(f)$ *at* $f_0\in\mathfrak{D}^+$ *towards* $g\in \boldsymbol{C}(S)$ is defined by

$$\lim_{\varepsilon\to 0}\frac{A(f_0+\varepsilon g)-A(f_0)}{\varepsilon},$$

if this limit exists. We denote it by $D_g A(f_0)$.

Definition 3.4. Consider, for $f\in\mathfrak{D}^+\cap\mathfrak{D}(\mathfrak{G}^0)$, the following equation

$$\frac{\partial A(t,f)}{\partial t}=D_{c(f)}A(t,f), \tag{3.5}$$

where

$$c(f)=\mathfrak{G}^0 f+k(\cdot)F[.,f].$$

We call (3.5) *forward equation* corresponding to $\{T_t^0, K, \pi\}$.

Theorem 3.4. *Suppose a B.M.P.* X_t *has the branching system and satisfies the Assumption* 1, *then if we put for* $\boldsymbol{x}\in \boldsymbol{S}-\{\Delta\}$,

$$A_{\boldsymbol{x}}(t,f)=\boldsymbol{T}_t\hat{f}(\boldsymbol{x}),\quad f\in\mathfrak{D}^+\cap\mathfrak{D}(\mathfrak{G}^0),$$

it is a solution of the forward equation corresponding to the system $\{T_t^0, K, \pi\}$ *of* X_t *with the initial condition*

$$A_{\boldsymbol{x}}(0+,f)=\hat{f}(\boldsymbol{x}).$$

3.5. Equation of the mean number of particles.

Definition 3.5. F*or* $f\in \boldsymbol{B}(S)$, *put*

$$\check{f}(\boldsymbol{x})=\begin{cases}\sum_{j=1}^{n} f(x_j), & \text{if } \boldsymbol{x}\in S^n,\ (x_1,\cdots,x_n)\in\boldsymbol{x},\\ 0, & \text{if } \boldsymbol{x}=\partial \text{ or } \Delta.\end{cases} \tag{3.6}$$

Then $\check{f}(\boldsymbol{x})$ is a measurable function on $\boldsymbol{S}$.

Definition 3.6. Put for non-negative $f\in \boldsymbol{B}(S)$,

$$G(x,f)=\int_{\boldsymbol{S}}\pi(x,d\boldsymbol{y})\check{f}(\boldsymbol{y})\hat{e}(\boldsymbol{y}),\text{[11]}$$

and consider an equation

$$u_t(x)=\frac{1}{e}T_t^0(ef)(x)+\frac{1}{e}\int_0^t\int_S K(x,ds,dy)G(y,u_{t-s}), \tag{3.7}$$

where $e(\boldsymbol{x})=P_{\boldsymbol{x}}[e_\Delta=+\infty]$ (e_Δ is the explosion time). We call (3.7) the (generalized) *equation of the mean number of particles* corresponding to $\{T_t^0, K, \pi\}$.

We introduce

Assumption 2. 1°) For $f\in \boldsymbol{C}(S)$, $G(.,f)\in \boldsymbol{C}(S)$, and $\|G(.,f)\|\leqq M\|f\|$, (M is a positive constant).

2°) $e(x)\in \boldsymbol{C}(S)$ and $e>0$ on S.

Definition 3.7. Under Assumptions 1 and 2, we put

$$\begin{aligned}\frac{\partial u_t}{\partial t}&=\frac{1}{e}\mathfrak{G}^0(eu_t)+\frac{1}{e}kG(.,u_t),\\ &=\frac{1}{e}\mathfrak{G}(eu_t)+\frac{1}{e}k[G(.,u_t)-eu_t]\end{aligned} \tag{3.8}$$

11) For $f=f^+-f^-$, we put $G(x,f)=G(x,f^+)-G(x,f^-)$ if the right hand side is definite.

and we call (3.8) the (*parabolic*) *equation of the mean number of particles.*

Theorem 3.5. *Suppose a B.M.P.* X_t *has the branching system and* $0<e(x)\leqq 1, x\in S$. *Put, for* $f\in \boldsymbol{B}(S)$ *and* $f\geqq 0$

$$H_t f(\boldsymbol{x})=\frac{1}{\hat{e}(\boldsymbol{x})}T_t(\hat{e}\check{f})(\boldsymbol{x}). \tag{3.9}$$

Then it satisfies

$$H_{t+s}f(x)=H_tH_sf(x),\ x\in S,$$

and, if $f\in \boldsymbol{C}(S)$, *then*

$$u_t(x)=H_tf(x)$$

is a solution of (3.7) *corresponding to the system* $\{T_t^0, K, \pi\}$ *of* X_t *with initial value* f.

Moreover if B.M.P. X_t *satisfies Assumptions* 1 *and* 2, *then* H_t *is a strongly continuous semi-group on* $\boldsymbol{C}(S)$ *with*

$$\|H_t\|\leqq e^{ct},\ (c \text{ is a positive constant}),$$

and if $ef\in\mathfrak{D}(\mathfrak{G}^0)$, *then* $eu_t\in\mathfrak{D}(\mathfrak{G}^0)$ *and* u_t *is a solution of* (3.8) *satisfying*

$$\|u_t-f\|\to 0\ (t\downarrow 0).$$

Remark. If $e\equiv 1$ and if we put $f\equiv 1$, then we have

$$H_t1(\boldsymbol{x})=E_{\boldsymbol{x}}\ [\text{the number of particles at } t], \tag{3.10}$$

which represents the mean number of particles at t.

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