56. A Duality Theorem for Locally Compact Groups. IV

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1. As a sequel of the previous articles [1]~[3], the present paper is devoted to prove the duality theorem which is same as shown in [3], for certain class of locally compact semi-direct product G of a separable closed abelian normal subgroup N and a closed subgroup K satisfying the assumptions 1~4. These class contains the motion group on $\boldsymbol{R}^n$, the n-dimensional inhomogeneous Lorentz group, and the transformation group of straight line.

We call an operator field $\boldsymbol{T}=\{T(D)\}$ over the set Ω_0 of all equivalence classes (representative $D=\{U_g^D, \mathfrak{H}^D\}$) of irreducible unitary representations of G *admissible* when

(1) $T(D)$ is a unitary operator in $\mathfrak{H}^D$ for any D in Ω_0.

(2) For any irreducible decomposition $\int D^\lambda d\nu(\lambda)$ of $D_1 \otimes D_2$ which is related by U,

$$U(T(D_1)\otimes T(D_2))U^{-1}=\int T(D^\lambda)d\nu(\lambda) .$$

The main proposition of this paper is as follows.

Proposition. *For any admissible operator field $\boldsymbol{T}$, there exists unique element g in G such that*

$$T(D)=U_g^D \quad \textit{for any } D \textit{ in } \Omega_0 .$$

2. [Assumption 1] *G is a regular semi-direct product in the sense of Mackey* [4].

Consider the dual group $\hat{N}$ of abelian group N, then g in G gives a transformation $g(\hat{n})$ on $\hat{N}$ defined by

$$\langle g(\hat{n}), n\rangle=\langle \hat{n}, g^{-1}ng\rangle ,$$

where brackets show ordinary dual relation between N and $\hat{N}$. We choose a representative $\hat{n}$ in given G-orbit L in $\hat{N}$, and let the isotropy group of $\hat{n}$ in G be $G(\hat{n})$, then $G(\hat{n})$ is a semi-direct product of N and a subgroup $K(\hat{n})$ in K.

For any irreducible unitary representation $\tau=\{W_k^\tau, \mathfrak{H}^\tau\}$ of $K(\hat{n})$ consider the representation $D(\hat{n}, \tau)$ of G induced by the representation $\{\langle \hat{n}, n\rangle W_k^\tau, \mathfrak{H}^\tau\}$ of $G(\hat{n})$ $(g=nk)$.

From Mackey's results ([4] Th. 14.1 and 2), $D(\hat{n}, \tau)$ is irreducible and determined by L and τ besides unitary equivalence, and arbitrary irreducible unitary representation of G is given in this form.

By the definition, $D(\hat{e}, \rho)(=D(\rho))$ is regarded as a representation $\rho=\{V^{\rho}, \mathfrak{H}^{\rho}\}$ of factor group $K\sim(G/N)$. And elements of the space $\mathfrak{H}(\hat{n}, \tau)$ of representation $D(\hat{n}, \tau)$ $(\hat{n}\neq\hat{e})$ are represented as $\mathfrak{H}^{\tau}$-valued functions on G satisfying

$$\boldsymbol{f}(nkg)=\langle\hat{n}, n\rangle W_k^{\tau}\boldsymbol{f}(g)\,, \qquad \text{for any } n\in N,\ k\in K\,.$$

[Assumption 2] *There exists an G-invariant open semi-group A in $\hat{N}$, such that*

(i) *for any $\hat{n}$ in A, $K(\hat{n})$ is a compact subgroup of K,*

(ii) *for any $\hat{n}_1$ in $\hat{N}$, there exists a $\hat{n}_2$ in A such that the set $\{k\colon \hat{n}_1+k(\hat{n}_2)\in A\}$ has positive Haar measure in K.*

When $\hat{n}_l$ is in A, the compactness of $K(\hat{n}_l)$ allows us to apply the decomposition theorem given by Mackey [4].

$$D(\hat{n}_1, \tau_1)\otimes\cdots\otimes D(\hat{n}_l, \tau_l)\sim\int_S D(\hat{n}_1, \cdots, \hat{n}_l; \tau_1, \cdots, \tau_l; \tilde{k})d\nu(\tilde{k})\,,$$

where $\tilde{k}=(k_1, \cdots, k_l)$ runs over the representatives in the space S of $(K(\hat{n}_1)\times\cdots\times K(\hat{n}_l), \tilde{K})$-double cosets $(\tilde{K}=\{(k,\cdots,k)\in K\times\cdots\times K\})$, and ν is a measure over S such that a double coset-wise set in $K\times\cdots\times K$ is a null set with respect to the Haar measure $\mu^l=\mu\times\cdots\times\mu$ if and only if its canonical image in S is a ν-null set. $D(\hat{n}_1, \cdots, \hat{n}_l; \tau_1, \cdots, \tau_l; \tilde{K})$ shows induced representation of G by the restriction of $\langle\sum_j^l k_j^{-1}(\hat{n}_j), n\rangle(k_1^{-1}(\tau_1)\otimes\cdots\otimes k_l^{-1}(\tau_l))$, to $N(k_1^{-1}K(\hat{n}_1)k_1\cap\cdots\cap k_l^{-1}K(\hat{n}_l)k_l)$. $(k_j^{-1}(\tau_j)=\{W^j_{k_jkk_j^{-1}}, \mathfrak{H}^{\tau_j}\}$: a representation of the group $k_j^{-1}K(\hat{n}_j)k_j)$.

The assumption 2 (ii) asserts the irreducible decomposition of $D(\hat{n}_1, \tau_1)\otimes D(\hat{n}_2, \tau_2)$ $(\hat{n}_2\in A)$ has a component which is a direct integral of $D(\hat{n}, \tau)$ $\hat{n}\in A$ with positive measure.

Moreover the corresponding vector in the space of representation on the right hand side to $\boldsymbol{f}_1\otimes\cdots\otimes\boldsymbol{f}_l$ in $\mathfrak{H}(\hat{n}_1, \tau_1)\otimes\cdots\otimes\mathfrak{H}(\hat{n}_l, \tau_l)$ $(\hat{n}_1, \cdots, \hat{n}_l\neq\hat{e})$ by this decomposition is the function $\boldsymbol{f}_1(k_1g)\otimes\cdots\otimes\boldsymbol{f}_l(k_lg)$ on G.

Evidently, $$D(\rho_1)\otimes D(\rho_2)\sim D(\rho_1\otimes\rho_2)\,.$$

And $D(\rho)\otimes D(\hat{n}, 1)\sim D(\hat{n}, \rho|_{K(\hat{n})})$, where the right hand side shows the induced representation of G by $\langle\hat{n}, n\rangle\rho|_{K(\hat{n})}(\rho|_{K(\hat{n})}$: the restriction of ρ to $K(\hat{n}))$ of the subgroup $NK(\hat{n})$, and the corresponding vector to $v\otimes f$ of $\mathfrak{H}^{\rho}\otimes\mathfrak{H}(\hat{n}, 1)$ is the function $f(g)(U_g^{\rho}v)$ on G. If $\rho|_{K(\hat{n})}\sim\sum_j\tau_j$ (τ_j: irreducible component with projection P_j), then $D(\rho)\otimes D(\hat{n}, 1)$ contains the component equivalent to $D(\hat{n}, \tau_j)$ and the component of above vector is given by $f(g)(P_jU_g^{\rho}v)$. Moreover, in the case of $\tau_j\sim 1$, we can set a $K(\hat{n})$-invariant vector φ in ${}_{\rho}\mathfrak{H}$ as $f(g)\langle U_g^{\rho}v, \varphi\rangle\varphi=f(g)P_jU_g^{\rho}v$ which corresponds to the function $f(g)\langle U_g^{\rho}v, \varphi\rangle$ in the space $\mathfrak{H}(\hat{n}, 1)$.

Lastly we set up the following assumptions.

[Assumption 3] *The duality theorem of the same type is true in the case of* K.

[Assumption 4] *There exists a finite set* $\hat{N}=\{\hat{n}_j\}$ $(1\leqq j\leqq l)$ *and a neighborfood* V *of* e *in* K *such that the map corresponding* $(k_1, \cdots, k_l)\in V\times \cdots \times V$ *to* $\sum k_j(\hat{n}_j)$ *is an open map.*

3. Now we are on the step to prove the main theorem. Let $\boldsymbol{T}=\{T(D)\}$ is a given admissible operator field. We can consider $\boldsymbol{T}$ as an admissible operator field on the dual space of K which is imbedded as a subset in Ω_0, assumption 3 assures existence of k_0 in K such that $T(D(\rho))=U_{k_0}^{D(\rho)}$ for any ρ. Define an admissible operator field $\boldsymbol{T}_0=\{T_0(D)\}$ by $\boldsymbol{T}_0=\boldsymbol{T}\boldsymbol{U}_{k_0}^{-1}$, then obviously $T_0(D(\rho))=I_{D(\rho)}$ (identity operator in $\mathfrak{H}^\rho$). And it is enough to show that $\boldsymbol{T}_0=\boldsymbol{U}_n$ for some $n\in N$. On the component of $D(\rho)\otimes D(\hat{n}, 1)$ which is equivalent to $D(\hat{n}, \tau_j)$ the admissibility of $\boldsymbol{T}_0$ gives,

$$T_0(D(\hat{n}, \tau_j))(f(g)P_j U_g^\rho v)= T_0(D(\hat{n}, 1)f(g))P_j U_g^\rho v\ , \tag{1}$$

and for the case of $\tau_j=1$,

$$\begin{aligned} T_0(D(\hat{n}, 1))(f(g)\langle U_g^\rho v, \varphi\rangle)&=\langle U_g^\rho T_0(D(\rho))v, \varphi\rangle(T_0(D(\hat{n}, 1))f)(g)\\ &=\langle U_g^\rho v, \varphi\rangle(T_0(D(\hat{n}, 1))f)(g). \end{aligned} \tag{2}$$

Because of ρ, v, f are arbitrary and from (2), $T_0(D(\hat{n}, 1))$ must be an operation to multiply a measurable function $c(\hat{n}, g)$ on G such that $|c(\hat{n}, g)|=1$, $c(\hat{n}, nkg)=c(\hat{n}, g)$ for $n\in N$, $k\in K(\hat{n})$. While (1) results $T_0(D(\hat{n}, \tau))$ is the operator of same form as $T_0(D(\hat{n}, 1))$ independently to τ. From the equivalence of $D(\hat{n}, 1)$ and $D(g(\hat{n}), 1)$, the function $c(\hat{n}, g)$ coincides with $c_0(g^{-1}(\hat{n}))$ for a function c_0 on $\hat{N}$ for almost all g.

For the determination of c_0, the decomposition of

$$D(\hat{n}_0, \tau_0)\otimes D(\hat{n}_1, \tau_1)\otimes \cdots \otimes D(\hat{n}_l, \tau_l) \qquad (\hat{n}_0, \hat{n}_1, \cdots, \hat{n}_l\in A)$$

is available. Simple argument leads us to that $D(\hat{n}_0,\hat{n}_1,\cdots,\hat{n}_l; \tau_0,\tau_1,\cdots, \tau_l; \tilde{k})(=D_1)$ is decomposed to a discrete direct sum of $D(\sum_{j=0}^{l} k_j^{-1}(\hat{n}_j), \tau)$. Since the operators $T_0(D(\sum k_j^{-1}(\hat{n}_j), \tau))$ are all same form for any τ, the operator $T_0(D_1)$ is represented as an operator to multiply the function $c_0(g^{-1}(\sum k_j^{-1}(\hat{n}_1))$. In the relation

$$\begin{aligned} (T_0(D(\hat{n}_0, \tau_0))\boldsymbol{f}_0)(k_0 g)\otimes \cdots \otimes(T_0(D(\hat{n}_l, \tau_l))\boldsymbol{f}_l)(k_l g)\\ =(T_0(D(\textstyle\sum k_j^{-1}(\hat{n}_j), \tau))\boldsymbol{f}_0(k_0 g)\otimes \cdots \otimes\boldsymbol{f}_l(k_l g))\ , \end{aligned}$$

we substitute the forms of operators and get

$$c_0(g^{-1}k_0^{-1}(\hat{n}_0))\times \cdots \times c_0(g^{-1}k_l^{-1}(\hat{n}_l))=c_0(g^{-1}(\textstyle\sum k_j^{-1}(\hat{n}_j))) \qquad (\text{a.a. } k_j, g).$$

Exclude g and calculate intergration

$$\begin{aligned} &\int_{V\times\cdots\times V} c_0\Big(\sum_{j=0} k_j^{-1}(\hat{n}_j)\Big)f(\tilde{k})d\mu^l(\tilde{k})\\ &\qquad =c_0(k_0^{-1}(\hat{n}_0))\int\prod_{j=1} c_0(k_j^{-1}(\hat{n}_j))f(\tilde{k})d\mu^l(\tilde{k}) \end{aligned}$$

for any continuous function f on the space $V\times \cdots \times V$ in the assumption 4. The left hand side is a continuous function of $\hat{n}_0$, so $c_0(k_0^{-1}(\hat{n}_0))$ too, consequently $c_0(\hat{n}_0)$ is continuous over A, and

$$c_0(\hat{n}_1+\hat{n}_2)=c_0(\hat{n}_1)c_0(\hat{n}_2) \qquad \text{for } \hat{n}_1,\ \hat{n}_2 \in A\ . \tag{3}$$

But the assumption 2(ii) means for any $\hat{n}$ in $\hat{N}$, there exists $\hat{n}_1$ in A such that $\hat{n}+\hat{n}_1=\hat{n}_2$ is in A. From (3), if we define $c_0(\hat{n})=c_0(\hat{n}_2)/c_0(\hat{n}_1)$, then c_0 is uniquely extendable as a character on $\hat{N}$. That is there exists a n in N and

$$c_0(\hat{n})=\langle \hat{n}, n\rangle\ .$$

Immediate calculation shows

$$T_0(\hat{n}, \tau)=U_n^{D(\hat{n},\tau)} \qquad \hat{n}\in A\ . \tag{4}$$

Again we apply the assumption 2(ii) to the decomposition of $D_1(\hat{n}_1, \tau_1)\otimes D_2(\hat{n}_2, \tau_2)$ ($\hat{n}_2 \in A$), substituting the above formula of $T_0(\hat{n}, \tau)$ on the component which is a direct integral of $D(\hat{n}, \tau)$'s ($\hat{n}\in A$), easily it is shown that the equation (4) is valid for any $\hat{n}\in\hat{N}$.

q.e.d.

References

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