55. *Skewness in Special Semi-Modular Lattices*

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In his lattice-theoretic formulation of incidence geometry, Gorn [2] uses a Condition E, to be called skewness [3] here, without considering its independence from the other axioms. This note is a partial answer to this question. Lattice-theoretic notions not specifically defined will be in agreement with those of Birkhoff [1].

Gorn's incidence geometry is a relatively complemented semimodular lattice of length greater than or equal to 5 and with a greatest lower bound 0. Let L with order $\leqq$ and operations $+, \cdot$ be such a lattice. The incidence geometry satisfies the further condition that it is *special* [3], i.e., if $a, b \in L$ such that $ab > 0$, then $(a, b)M$ ((a, b) is a modular pair). Let L be special. L satisfies *skewness* means that if $l_1, l_2, l_3, l_4 \in L$ are lines (elements of L covering points) such that $l_i + l_j$ covers l_i and l_j for $(i, j) = (1, 2), (1, 3), (2, 3), (2, 4), (3, 4)$, then $l_1 + l_4$ covers l_1 and l_4.

Theorem. *If L is of length greater than* 5, *then L satisfies skewness.*

Proof. Let $l_1, l_2, l_3, l_4 \in L$ be lines such that $l_i + l_j$ covers l_i and l_j for $(i, j) = (1, 2), (1, 3), (2, 3), (2, 4), (3, 4)$. Suppose $l_1 + l_4$ does not cover l_1 or l_4, i.e., l_1 and l_4 are skew lines (see [2], p. 164). Define $h = l_1 + l_4$. Then the closed interval from 0 to h is of length 5 and h covers $l_i + l_j$ for $(i, j) = (1, 2), (1, 3), (2, 3), (2, 4), (3, 4)$. Let p be a point such that $p \nleqq h$, which exists since L is relatively complemented and of length greater than 5. Then $p + h$ covers h. Define $m_i = p + l_i$ for each i. Then the m_i are distinct, m_i covers l_i for each i, $m_2 + m_3$ covers m_2 and m_3, and $(m_1, m_4)M$ and $(m_2, m_3)M$ since $m_1 m_4,\ m_2 m_3 \geqq p > 0$. Further, $m_1 + m_4 = (p + l_1) + (p + l_4) = p + (l_1 + l_4) = p + h$. If $m_2 m_3 > p$, then $m_1 \cdot m_4 = (m_1 + m_2)(m_1 + m_3) \cdot (m_2 + m_4)(m_3 + m_4) = (m_1 + m_2)(m_2 + m_4) \cdot (m_1 + m_3)(m_3 + m_4) = m_2 \cdot m_3 > p$. Thus m_1, m_4 cover $m_1 m_4$, but $m_1 + m_4 = p + h$ does not cover m_1, m_4 contrary to $(m_1, m_4)M$. Hence $m_2 m_3 = p$. Now m_2, m_3 do not cover $m_2 m_3$ while $m_2 + m_3$ covers m_2 and m_3. This is contrary to $(m_2, m_3)M$. Thus, necessarily, $l_1 + l_4$ covers l_1 and l_4.

References

[1] Garrett Birkhoff: Lattice Theory (rev. ed.). Amer. Math. Soc. Colloquium Publications. Vol. 25 (1948).

[2] Saul Gorn: On incidence geometry. Bull. Amer. Math. Soc., **46**, 158–167 (1940).

[3] L. R. Wilcox: Modular extensions of semi-modular lattices. Bull. Amer. Math. Soc., **61**, 524 (1955).