53. *Modular Extensions of Point-Modular Lattices*

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1. **Introduction.** It is well known that a linear space can be represented using properties of the lattice Λ of all subspaces. Properties of certain lattice subsets of Λ are used here to realize this representation. These subsets are designated as point-closed subsystems and a point-modular point-complemented irreducible lattice of length$\geqq 4$ is shown to be a characterization of such a point-closed subsystem. Then a point-modular point-complemented lattice L is decomposed into a subdirect union of point-modular point-complemented irreducible lattices and in the case that L is complete into the direct union. This generalizes the classical representation theory for lattices of the type Λ.

In an application of these results the family of finite dimensional subspaces of a linear space, being a point-closed subsystem of the lattice of all subspaces, is characterized lattice-theoretically. A second application is made to the linear systems of G. W. Mackey [5] for which he states conditions that a family of subspaces of a (real) linear space be the family of closed subspaces relative to some regular linear system constructed on the linear space. Such a family forms a point-closed subsystem and can be used in the description of the linear space. Thus the family of closed subspaces of a regular linear system is characterized lattice-theoretically without the linear space being given explicitly.

The lattice-theoretic notions not defined here or for which no reference is given are in agreement with those of Birkhoff [2]. Let the system $(L, +, \cdot)$ be a lattice. For $S \subset L$ and $b, c \in L$, $(b, c)M_S$ (read (b, c) *modular relative to* S) means $(a+b)c = a+bc$ for every $a \in S$ such that $a \leqq c$. For $a, b \in L$, $a \succ b$ (dually, $b \prec a$) is written for a covers b. The notations $\vee$ and $\wedge$ are set-theoretic union and intersection.

2. **Point-modularity and point-complementation.** In this section $(L, +, \cdot)$ is a lattice with zero 0 and P the set of points in L. The relation M_P is called *point-modularity* and L is said to be point-modular if $M_P = L \times L$. Also L is said to be *point-complemented* if for $a, b \in L$ such that $a < b$ there exists $p \succ 0$ such that $p \leqq b$, $p \nleqq a$. The following brief development is used later.

Lemma 1. *If L is point-complemented and $b, c \in L$ such that*

$(b, c)M_P$, $b \prec b+c$ then $bc \prec c$.

Proof. Let $b \nleqq c$. Suppose $bc < a < c$. Then $bc = ba$ and $b+c = b+a$. Let $p \succ 0$ such that $p \leqq a$, $p \nleqq ba$. Then $p \nleqq b$ and $b \leqq p+b \leqq b+c$, whence $p+b = b+c$ since $b = p+b$ implies $p \leqq b$. Now $(p+b)c = (b+c)c = c > a \geqq p+bc$ contrary to $(b, c)M_P$. Thus $bc \prec c$.

It now can be shown that for L point-modular and point-complemented the Jordan-Dedekind chain condition holds. (This is done in [3] with the dual condition: $bc \prec c$ implies $b \prec b+c$.) This generalizes results of K. Menger [7] for which he uses relative complementation. The latter (with atomicity) is stronger than point-complementation and too strong for the application made to linear systems in Section 6. Further, for L point-complemented and of finite length, L is modular if and only if it is point-modular.

3. Point-closed subsystems. In this section $(\Lambda, \cup, \cap)$ is a complete complemented modular atomic lattice with zero 0 and unit 1 which satisfies

(1) if $A \subset \Lambda$, $p \succ 0$ such that $p \leqq \bigcup A$ then $p \leqq \bigcup B$ for some finite $B \subset A$.

Two properties of the lattice of all subspaces of a linear space (of dimension $\geqq 3$) introduced later but not needed here are irreducibility [4, p. 453] and length $\geqq 4$.

A set $L \subset \Lambda$ is said to be a *point-closed subsystem* of Λ if (a) $0 \in L$; (b) for $a, b \in L$, $a \cap b \in L$ (write $a \cap b = ab$); (c) for $a, b \in L$, l.u.b. $\{a, b\}$ exists with respect to the elements of L (write l.u.b. $\{a, b\} = a+b$); (d) for $a \in L$ and $p \succ 0$, $p \cup a \in L$. In the remainder of this section L is to be a point-closed subsystem of Λ. Then all the elements of Λ of finite dimension are in L.

Theorem 1. *The system $(L, +, \cdot)$ is a point-modular point-complemented lattice.*

Proof. It is immediate that the system is a lattice. For the point-modularity let $b, c, p \in L$ with $0 \prec p \leqq c$. Then

$$(p+b)c = (p \cup b) \cap c = p \cup (b \cap c) = p+bc ,$$

whence $(b, c)M_P$. The point-complementation follows from the complementation and atomicity of Λ.

For $a \in \Lambda$ define $F(a) = \{x \in L : x \leqq a, \dim x < \infty\}$. Then for $a \in \Lambda$, $a = \bigcup F(a)$. (This is essentially a theorem of Orrin Frink, Jr. [4, Theorem 8].) Thus $\Lambda = \{\bigcup F(a) : a \in \Lambda\}$. It is in this sense that Λ is considered to be generated by L. To free the discussion from elements of $\Lambda - L$ it is noted that $F(a)$ is an ideal in L that contains only elements of finite dimension, and conversely, it is proved in the next theorem that such an ideal is a set of the type $F(a)$. These ideals are used when imbedding, as a point-closed subsystem, an abstractly described lattice L into a lattice of the type Λ.

Theorem 2. *If A is an ideal in L that contains only elements of finite dimension then $A=F(a)$ for some $a\in \Lambda$.*

Proof. Define $a=\bigcup A$. Certainly, $A\subset F(a)$. Let $y\in F(a)$. Then $y\leqq \bigcup A$. Since $\dim y<\infty$, there exists finite $B\subset A$ such that $y\leqq \bigcup B$ by virtue of (1). Hence $y\in A$ and $F(a)\subset A$.

4. The general representation theory. The lattice L is now described abstractly using properties of the last section and is shown to be contained (isomorphically) as a point-closed subsystem in a lattice of the type Λ. Throughout this section $(L, +, \cdot)$ is a point-modular point-complemented lattice with zero 0.

For $a\in L$ define $P(a)=\{p\in L\colon p\succ 0, p\leqq a\}$. Then for $a\in L, a=\sum P(a)$ because $a\nleqq b$ for any other upper bound b of $P(a)$ implies existence of $p\succ 0$ such that $p\nleqq b$ and $p\leqq a$, a contradiction. (This is a generalization of the previously mentioned theorem of Frink.) In light of the results and comments of Section 2 it is meaningful to use the notion of the dimension of an element.

For $T\subset L$ define $[T]$ to be the smallest ideal containing T. Also define $I=\{a\in L\colon \dim a<\infty\}$ and $\phi=\{0\}$. Then $I=[\{p\in L\colon p\succ 0\}]$. Define $\mathcal{L}$ to be the set of all ideals of L that are subsets of I and $\mathcal{S}=\{[P(a)]\colon a\in L\}$. For $\alpha, \beta\in\mathcal{L}$ define $\alpha\leqq\beta$ to mean $\alpha\subset\beta, \alpha\cup\beta=[\alpha\vee\beta], \alpha\cap\beta=\alpha\wedge\beta$. It is immediate that $(\mathcal{L}, \cup, \cap)$ is a complete lattice with zero ϕ and unit I; for $\alpha\in\mathcal{L}, \alpha=[\{p\in\alpha\colon p\succ 0\}]$; and $\mathcal{S}\subset\mathcal{L}$.

Lemma 2. *The lattice $\mathcal{L}$ is complemented, modular, atomic, and satisfies* (1).

Proof. For the complementation let $\alpha\in\mathcal{L}$. An application of Zorn's Lemma yields $\beta\in\mathcal{L}$ such that $\alpha\cap\beta=\phi$ and $\beta'>\beta$ implies $\alpha\cap\beta'\neq\phi$. To show $\alpha\cup\beta=I$ let $p\succ 0$. Also let $p\notin\beta$; otherwise, $p\in[\alpha\vee\beta]$. Now $\beta<\beta\cup[P(p)]$. Thus $\alpha\cap(\beta\cup[P(p)])\neq\phi$, whence there exists $y\in\alpha$ such that $y\neq 0, y\leqq b+p$ for some $b\in\beta$. Suppose $y+b<b+p$. Then $b=y+b$ since $b\prec b+p$, whence $y\leqq b$. Thus $y\in\alpha\cap\beta$ contrary to $y\neq 0$. Hence $y+b=b+p\geqq p$ and $p\in[\alpha\vee\beta]$. Thus $I\leqq[\alpha\vee\beta]=\alpha\cup\beta$.

The proof of the modularity is similar to that for the lattice of all ideals in a modular lattice. The point-modularity is sufficient in the place of modularity since the ideals considered here contain only elements of finite dimension.

The atomicity follows from the point-complementation.

Finally, for (1) let $\alpha_j\in\mathcal{L}$ for $j\in\mathcal{J}$ and π be a point in $\mathcal{L}$ such that $\pi\leqq\bigcup_{\mathcal{J}}\alpha_j$. Define $\beta=\{x\in L\colon x\in\bigcup_{\mathcal{N}}\alpha_j$ for some finite $\mathcal{N}\subset\mathcal{J}\}$. Obviously, $\beta\subset\bigcup_{\mathcal{J}}\alpha_j$. The set β is an ideal containing α_j for every

$j \in \mathcal{J}$. Thus $\bigcup_{\mathcal{J}} \alpha_j \subset \beta$ and $\beta = \bigcup_{\mathcal{J}} \alpha_j$. Let $\pi = [P(p)]$ for some $p > 0$. Then $p \in \beta$, whence $p \in \bigcup_{\mathcal{K}} \alpha_j$ for some finite $\mathcal{K} \subset \mathcal{J}$, i.e., $\pi \leqq \bigcup_{\mathcal{K}} \alpha_j$. This completes the proof of the lemma.

Lemma 3. *The set $\mathcal{S}$ is a point-closed subsystem of $\mathcal{L}$ and L is isomorphic to $\mathcal{S}$ under $a \to [P(a)]$.*

Proof. The first part is immediate. For the remainder let $a, b \in L$. If $a \leqq b$ then $P(a) \subset P(b)$, whence $[P(a)] \leqq [P(b)]$. Conversely, let $[P(a)] \leqq [P(b)]$. Also let $p \in P(a)$; otherwise, if $P(a)$ is empty, $a = 0 \leqq b$. Now $p \in [P(a)]$, whence $p \in [P(b)]$. This implies $p \leqq b$; thus $p \in P(b)$. Hence $P(a) \subset P(b)$. Now $a = \sum P(a) \leqq \sum P(b) = b$. Thus the mapping is an isomorphism.

These results are summarized in the following theorem which is similar to an imbedding theorem of Frink [4, Theorem 14] in that both extension lattices are of the same type. However, he imbeds a complemented modular lattice (not necessarily atomic) while here the imbedded lattice is point-modular and point-complemented (necessarily atomic but not necessarily complemented and modular).

Theorem 3. *The lattice L is isomorphic to a point-closed subsystem of a lattice of the type Λ of Section* 3; *Λ is unique up to isomorphism.*

If the lattice Λ of Section 3 is in addition irreducible and of length$\geqq 4$ then every point-closed subsystem is. Conversely, if the lattice L of the present section is irreducible and of length$\geqq 4$ then its extension $\mathcal{L}$ is. Reinhold Baer [1] shows that such a Λ is isomorphic to the lattice of all subspaces of a linear space, the latter being unique up to isomorphism. These comments and Theorem 3 yield the following representation theorem.

Theorem 4. *If L is a point-modular point-complemented irreducible lattice of length$\geqq 4$ then L is isomorphic to a point-closed subsystem of the lattice of all subspaces of some linear space, the latter being unique up to isomorphism.*

A decomposition into irreducible lattices is now given. Comments similar to those preceding Theorem 3 can be made about the comparison of these results with Theorem 15 of Frink [4]. (See [4, p. 466] for the meaning of subdirect union.) The second part of Theorem 5 is a generalization of a part of Theorem 2 of J. E. McLaughlin [6]. Finally, Theorems 4 and 5 give a generalization beyond that of McLaughlin [6] of the classical representation theory for complete complemented modular atomic lattices satisfying (1).

Theorem 5. *If L is a point-modular point-complemented lattice then L is isomorphic to a subdirect union of a family of point-modular point-complemented irreducible lattices. Moreover, L is* *isomorphic to the direct union and each member of the family is complete if L is complete.*

Proof. Let $\mathcal{L}$ be isomorphic to $\times_{i\in\mathcal{J}}\mathcal{L}_i$, the direct union of irreducible lattices of the type Λ of Section 3 [4, pp. 453–456]. For $i\in\mathcal{J}$ define $\mu_i=\bigcup\mathcal{L}_i$ and $\mathcal{S}_i=\{\mu_i\cap\alpha\colon\alpha\in\mathcal{S}\}$. Then $\mathcal{S}_i$ is a point-closed subsystem of $\mathcal{L}_i$, whence each $\mathcal{S}_i$ is point-modular, point-complemented and irreducible. Define T, a mapping of $\mathcal{S}$ into $\times_{i\in\mathcal{J}}\mathcal{S}_i$, as follows: for $\alpha\in\mathcal{S}$, $\alpha^T=(\mu_i\cap\alpha\colon i\in\mathcal{J})$. T is easily seen to be an isomorphism between $\mathcal{S}$ and a sublattice of $\times_{i\in\mathcal{J}}\mathcal{S}_i$. Since L and $\mathcal{S}$ are isomorphic, the proof of the first part is complete.

Let L be complete. Then $\mathcal{S}$ is complete; hence each $\mathcal{S}_i$ is. To show that T is onto let $\tau_i\in\mathcal{S}_i$ and define $\alpha=\bigcup_{i\in\mathcal{J}}\tau_i\in\mathcal{L}$. Then $\alpha=\bigcup\{\pi\in\mathcal{L}\colon\phi\prec\pi\leqq\tau_i \text{ for some } i\in\mathcal{J}\}\in\mathcal{S}$. The proof of $\tau_i=\mu_i\cap\alpha$ follows that of Theorem 2 of McLaughlin [6]. This completes the proof.

5. Family of finite dimensional subspaces. The family of finite dimensional subspaces of a linear space can be readily shown to be a point-closed subsystem of the lattice of all subspaces. Thus if the lattice L of Theorem 4 satisfies in addition the descending chain condition, it follows that L is isomorphic to the particular point-closed subsystem consisting of the finite dimensional subspaces of some linear space. Again the latter is unique up to isomorphism.

6. Linear systems. The linear systems of G. W. Mackey [5] can be considered over arbitrary division rings, not necessarily the real field. The theory is essentially the same and includes the following relevant properties: if S and T are closed subspaces of a linear system and x an element of the linear space then $S\wedge T$ and the subspace spanned by $S\vee\{x\}$ are closed; the zero dimensional subspace of a linear system is closed if and only if the linear system is regular. Thus the family of closed subspaces of a regular linear system is a point-closed subsystem of the lattice of all subspaces.

Further, a family $\mathcal{S}$ of subspaces of a linear space X is the family of closed subspaces relative to some linear system constructed on X if and only if it satisfies the following: (a) for $\mathcal{T}\subset\mathcal{S}$, $\bigwedge\mathcal{T}\in\mathcal{S}$; (b) for $S\in\mathcal{S}$ and $x\in X$, the subspace spanned by $S\vee\{x\}$ is in $\mathcal{S}$; (c) every $S\in\mathcal{S}$ such that $S\neq X$ is an intersection of members of $\mathcal{S}$ which are hyperplanes in the lattice of all subspaces. (A hyperplane in the lattice of all subspaces of a linear space is a subspace with deficiency one.) Finally, two regular linear systems are isomorphic if and only if their lattices of closed subspaces are isomorphic.

These properties and Theorem 4 yield the following representation theorem which is the same as the second part of Theorem 2 of McLaughlin [6]; the hypotheses are formally different but easily seen to be equivalent. However, the objectives leading to the statements of the theorem and the techniques employed in its proofs are quite different.

Theorem 6. *If L is a complete point-modular irreducible lattice (with zero 0 and unit 1) of length$\geqq 4$ which satisfies*

(2) *for* $a \in L, \sum \{p \in L: p \leqq a, p \succ 0\} = \prod \{h \in L: h \geqq a, h \prec 1\}$,

then L is isomorphic to the lattice of closed subspaces of a regular linear system, the latter being unique up to isomorphism.

Proof. It is a consequence of (2) that for

$$a \in L, \sum \{p \in L: p \leqq a, p \succ 0\} = a = \prod \{h \in L: h \geqq a, h \prec 1\}.$$

This is equivalent to the combined conditions that L is point-complemented and for $a \in L$ with $a \neq 1$, a is the g.l.b. of some set of hyperplanes of L. It now follows that L is isomorphic to a point-closed subsystem $\mathcal{S}$ of the lattice of all subspaces of some linear space X; this $\mathcal{S}$ is the family of closed subspaces of some linear system constructed on X; this linear system is regular.

References

[1] R. Baer: Linear Algebra and Projective Geometry. Academic Press, New York (1952).

[2] G. Birkhoff: Lattice Theory (rev. ed.). Amer. Math. Soc. Colloquium Publications. Vol. 25 (1948).

[3] M. L. Dubreil-Jacotin, R. Croisot, and R. Lesieur: Leçons sur la théorie des treillis, des structures algébriques ordonnées et des treillis géométrique. Paris (1953).

[4] Orrin Frink, Jr.: Complemented modular lattices and projective spaces of infinite dimension. Trans. Amer. Math. Soc., **60**, 452-467 (1946).

[5] G. W. Mackey: On infinite-dimensional linear spaces. Trans. Amer. Math. Soc., **57**, 155-207 (1945).

[6] J. E. McLaughlin: The normal completion of a complemented modular point lattice. Proc. of Symposia in Pure Math. of Amer. Math. Soc., **2**, 78-80 (1961).

[7] Karl Menger: New foundations of projective and affine geometry. Ann. Math., **37**, 456-482 (1936).