50. *On Griss Algebra.* I

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About fifteen years ago, the late Professor G. F. C. Griss introduced and developed the logic of negationless intuitionistic mathematics and its mathematics (see [4], [5]). The logic is different from the two valued classical logic and the intuitionistic logic by L. E. J. Brouwer and A. Heyting (see [2], [3], and [6]). In the negationless mathematics, G. F. C. Griss rejected, in general, the notion of disjunction form. On the other hand, N. Dequoy [1] considered a projective geometry from standpoints of negationless logic and mathematics. In this note, we shall give an algebraic formulation of the negationless logic and define Griss algebra.

1. An algebraic formulation of the negationless logic. In this section, we shall take up the following formulation. Let $\boldsymbol{G}=\langle X, 0, \vee, *\rangle$ be an algebra with two binary operations $\vee, *$ on a set X satisfying the following conditions:

(1) $x\vee x\leqslant x$,

(2) $x\vee y\leqslant y\vee x$,

(3) $(x\vee y)*(x\vee z)\leqslant y*z$,

(4) $x*y\leqslant(x*z)\vee(z*y)$,

(5) $x\leqslant y\vee x$,

(6) $0\leqslant x$,

(7) if $x\leqslant y$ and $y\leqslant x$, then $x=y$,

where $x\leqslant y$ is defined by $x*y=0$.

Hence $0\leqslant x$ is equivalent to $0*x=0$. Of course we can define the dual algebra of the algebra X, but we do not consider it. We first deduce some lemmas.

Axioms (1) and (5) imply

(8) $x\vee x=x$.

Hence we have

(9) $0\vee 0=0$.

From (2), we have

(10) $x\vee y=y\vee x$.

By axiom (4), $x*x\leqslant(x*(x\vee x))*((x\vee x)*x)=0$, hence

(11) $x\leqslant x$.

From axiom (4), $x*z\leqslant(x*y)\vee(y*z)$. If $x\leqslant y, y\leqslant z$, then $x\leqslant z$.

(12) $x\leqslant y, y\leqslant z$ imply $x\leqslant z$.

By (5) and (10), we have $x\leqslant 0\vee x=x\vee 0$. On the other hand, by axiom (3),

$$(x \vee 0) * (x \vee x) \leqslant 0 * x = 0.$$

Hence $(x \vee 0) * x = 0$, then $x \vee 0 \leqslant x$. This shows

(13) $x \vee 0 = 0 \vee x = x$.

By (3) and (10), we have

$$(y \vee x) * (z \vee x) \leqslant y * z.$$

If $x \leqslant y$, then $x * y = 0$, and

$$(x \vee y) * y = (x \vee y) * (y \vee y) \leqslant x * y = 0.$$

Hence $x \vee y \leqslant y \leqslant x \vee y$. Therefore

(14) $x \leqslant y$ implies $x \vee y = y$.

If $x, y \leqslant z$, then $x \vee z = z$, and by axiom 4, we have

$$(x \vee y) * z = (x \vee y) * (x \vee z) \leqslant y * z = 0.$$

Therefore, $x \vee y$ is the supremum of x and y, hence the algebra $\boldsymbol{G}$ is a semilattice on $\vee$ (for semilattices, see G. Szasz [8] p. 47). This means that $\boldsymbol{G}$ satisfies the following conditions:

a) $x \vee x = x$,

b) $x \vee y = y \vee x$,

c) $x \vee (y \vee z) = (x \vee y) \vee z$.

Further $\boldsymbol{G}$ has the least element 0.

By (4) and (13), we have the following

(15) $x \leqslant y$ implies $z * y \leqslant z * x$ and $x * z \leqslant y * z$.

Next we shall prove some lemmas deduced by G. F. C. Griss [4]. For the proofs, we freely use results above.

(16) $(x \vee y) * (z \vee u) \leqslant (y * u) \vee (x * z)$,

$$(x \vee y) * (z \vee u) \leqslant (x * u) \vee (y * z).$$

Consider axiom (3): $(x \vee u) * (z \vee u) \leqslant x * z$, then we have

$$(y * u) \vee ((x \vee u) * (z \vee u)) \leqslant (y * u) \vee (x * z).$$

On the other hand, by axiom (3),

$$(x \vee y) * (x \vee u) \leqslant y * u,$$

so we have

$$((x \vee y) * (x \vee u)) \vee ((x \vee u) * (z \vee u)) \leqslant (y * u) \vee (x * z).$$

Applying axiom (4), we have

$$(x \vee y) * (z \vee u) \leqslant (y * u) \vee (x * z).$$

Hence we have (16). The second formula is clear and put $u = z$ in the first formula, then

(17) $(x \vee y) * z \leqslant (x * z) \vee (y * z)$.

By (15) and $x \leqslant x \vee y$, we have $x * z \leqslant (x \vee y) * z$. Similarly we have $y * z \leqslant (x \vee y) * z$. Hence

(18) $(x * z) \vee (y * z) \leqslant (x \vee y) * z$.

Therefore, by (17) and (18), we have the following

(19) $(x \vee y) * z = (x * z) \vee (y * z)$.

In the second section, we shall define a Griss algebra by using the results above.

2. Griss algebra. Let G be an algebra $\langle X, 0, \vee, *\rangle$ with two binary operations $\vee, *$ defined over X satisfying the following conditions:

(1) G is a semilattice on $\vee$, and contains 0 as the zero element (the least element).

(2) $x*x=0$ for every $x \in X$.

(3) $x \leqslant y$ (i.e., $x \vee y=y$) implies $z*y \leqslant z*x$ for every element $z \in X$.

(4) $(x \vee y)*z=(x*z)\vee(y*z)$ holds for all $x, y, z \in X$.

Then the algebra G is called a *Griss algebra*.

If $x \leqslant y$, then $x \vee y=y$, by axiom (4), we have

$$y*z=(x \vee y)*z=(x*z)\vee(y*z),$$

hence $x*z \leqslant y*z$.

(5) $x \leqslant y$ implies $x*z \leqslant y*z$.

Let $x \leqslant y$, then by (5),

$$0 \leqslant x*y \leqslant y*y=0,$$

hence $x \leqslant y$ implies $x*y=0$.

(6) If $x \leqslant y$, then $x*y=0$.

It is easily verified that (1), (2), (5), (6), and (7) in the first section hold in any Griss algebra G. Further we can verify axiom (4) in the first section: $(x \vee y)*(x \vee z) \leqslant y*z$. Consider (4) and substitute $x \vee z$ for z, then by $x*(x \vee z)=0$, we have

$$(x \vee y)*(x \vee z)=(x*(x \vee z))\vee(y*(x \vee z))=y*(x \vee z).$$

From axiom 3, $z \leqslant x \vee z$ implies $y*(x \vee z) \leqslant y*z$, hence

(7) $(x \vee y)*(x \vee z) \leqslant y*z$.

3. Ideals of Griss algebra. As already said, any Griss algebra G is a semilattice, so we can consider ideals of G. A subset I of G is called an *ideal* of G, if 1) for any $x, y \in I$, then $x \vee y \in I$ and 2) $x \in I$ and $y \leqslant x$ imply $y \in I$. The ideal theory of implicative semilattices algebraically formulated from the intuitionistic propositional calculus was recently developed by W. C. Nemitz [7]. Some of his results hold in the case of the Griss algebra.

Let G_1, G_2 be two Griss algebras. Consider a mapping h from G_1 onto G_2 satisfying

$$h(x*y)=h(x)*h(y),$$

then h is called a $*$-(onto) *homomorphism* (implicational homomorphism).

For any $*$-(onto) homomorphism, we have $h(0)=h(0*y)=h(0)*h(0)=0$, hence $h(0)=0$. Let $x \leqslant y$, then $x*y=0$ by (6) in the section 2, hence

$$h(0)=h(x*y)=h(x)*h(y).$$

This means $h(x) \leqslant h(y)$, i.e., h is isotone.

(1) *A $*$-(onto) homomorphism h is isotone and* $h(0)=0$.

If a $*$-homomorphism h is a $\vee$-homomorphism: $h(x \vee y)=h(x)\vee h(y)$, then $h^{-1}(0)$ is clearly an ideal of G_1.

Conversely, let h be a $*$-homomorphism that $h^{-1}(0)$ is an ideal of G_1, then h is isotone by (1). Hence $h(x)\vee h(y)\leqslant h(x\vee y)$. Let $h(x)$, $h(y)\leqslant u$, then we can find an element z of G_1 such that $h(z)=u$. Hence $h(x)\leqslant h(z)$ and $0=h(x)*h(z)=h(x*z)$. Therefore $x*z\in h^{-1}(0)$, similarly $y*z\in h^{-1}(0)$. By axiom (4) of Griss algebra, $(x\vee y)*z=(x*z)\vee(y*z)\in h^{-1}(0)$. Hence $0=h((x\vee y)*z)=h(x\vee y)*h(z)$, and so $h(x\vee y)\leqslant u$. This means $h(x\vee y)\leqslant h(x)\vee h(y)$.

(2) *$h^{-1}(0)$ is an ideal if and only if h is a $*$-homomorphism.*

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