48. On Propagation of Regularity in Space-variables for the Solutions of Differential Equations with Constant Coefficients

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Introduction. Let $P(D_t, D_x)$ be a differential operator with constant coefficients for which the plane: $t=0$ is characteristic. In the note [4] K. Shinkai and the author characterized this operator P through the Gevrey class $G(\alpha)\ (-\infty\leqq\alpha<1)$, with respect to space-variables, in which null solutions[1] of $Pu=0$ are able to exist.

In this note we are concerned with the converse problem: 'Is it possible to construct a null solution such that its derivative of some order has the discontinuity with respect to space-variables at some point $(t_0, x_0)\ (t_0>0)$?' Here we give a negative answer for this problem in the sense of Theorem 1. For example, the solutions of the wave equation $(\partial^2/\partial t\,\partial x)u=0$ have the form $u(t, x)=f(t)+g(x)$. Hence, if a solution of $(\partial^2/\partial t\,\partial x)u=0$ is analytic in x for negative t, then, necessarily, it is analytic in x for positive t. But, in order to generalize this phenomena, it is necessary to discuss the propagation of regularily, which has been studied by F. John [3], B. Malgrange [5], L. Hörmander [2], and J. Boman [1], with respect to only the space-variables. We shall use L^1-estimates according to J. Boman. The details will be published in the Funkcialaj Ekvacioj.

§1. Notations and preliminary lemmas. Let $(t, x)=(t, x_1, \cdots, x_\nu)$ be a point in the Euclidean $(1+\nu)$-space $R^{1+\nu}$, $\xi=(\xi_1, \cdots, \xi_\nu)$ be a point in the dual space E^ν of R^ν, and $\alpha=(\alpha_1, \cdots, \alpha_\nu)$ be a real vector whose elements are non-negative integers. We shall use notations:

$$(D_t, D_x)=(D_t, D_{x_1}, \cdots, D_{x_\nu})=(-i\partial/\partial t, -i\partial/\partial x_1, \cdots, -i\partial/\partial x_\nu),$$
$$|\alpha|=\alpha_1+\cdots+\alpha_\nu,\ \alpha!=\alpha_1!\cdots\alpha_\nu!,\ x\cdot\xi=x_1\xi_1+\cdots+x_\nu\xi_\nu,$$
$$D_x^\alpha=D_{x_1}^{\alpha_1}\cdots D_{x_\nu}^{\alpha_\nu},\ \xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_\nu^{\alpha_\nu}.$$

For a function $v(x)\in C_0^\infty(R^\nu)$ we define the Fourier transform $\tilde{v}(\xi)$ by

$$\tilde{v}(\xi)=\frac{1}{\sqrt{2\pi}^\nu}\int_{R^\nu}e^{-ix\cdot\xi}v(x)dx$$

1) A C^∞-solution u of $Pu=0$ is called a null solution, if $u\equiv0$ for $t\leqq0$ and $u\not\equiv0$ for $t>0$.

and for a function $u(t, x) \in C_0(R^{1+\nu})$ define the partial Fourier transform $\tilde{u}(t, \xi)$ by

$$\tilde{u}(t, \xi)=\frac{1}{\sqrt{2\pi}^{\nu}}\int_{R^\nu} e^{-ix\cdot\xi}u(t, x)dx.$$

Lemma 1. *Let $P(\lambda, \xi)$ be a differential polynomial of the form*

$$(1) \qquad P(\lambda, \xi)=Q_m(\xi)\lambda^m+Q_{m-1}(\xi)\lambda^{m-1}+\cdots+Q_0(\xi) \qquad (m\geqq 1,\ Q_m(\xi)\neq 0).$$

Then, for any real number a, b and positive function $\gamma(\xi)$ we have

$$(2) \qquad \begin{aligned}&\int_{E^\nu}\gamma(\xi)^{a-bt_0}\,|\,Q_m(\xi)\tilde{u}(t_0, \xi)\,|\,d\xi\\ &\leqq T_0^{m-1}\int_0^{T_0}\int_{E^\nu}\gamma(\xi)^{a-bt}P(D_t, \xi)\tilde{u}(t, \xi)\,d\xi dt\end{aligned}$$

$$(T_0>0,\ t_0\in(0, T_0),\ u\in C_0^\infty \text{ in } (0, T_0)\times R^\nu).$$

Proof. For a function $f(t)\in C_0^\infty$ in $(0, T_0)$, a complex number λ and real numbers μ, η, we set $g(t)=e^{\mu-i\lambda t}f(t)$, then $D_tg(t)=e^{\mu-i\lambda t}(D_t-\lambda)f(t)$. Then, we have

$$\begin{aligned}&e^{\mu-\eta t_0+(\Im m\lambda+\eta)t_0}|\,f(t_0)\,|=|\,g(t_0)\,|\\ &\leqq \operatorname{Min}\left\{\int_0^{t_0}|\,D_tg(t)\,|dt,\ \int_0^{t_0}|\,D_tg(t)\,|dt\right\}\\ &\leqq \operatorname{Min}\left\{\int_0^{t_0}e^{\mu-\eta t+(\Im m\lambda+\eta)t}|(D_t-x)f(t)|dt,\ \int_{t_0}^{T_0}e^{\mu-\eta t+(\Im m\lambda+\eta)t}|(D_t-\lambda)f(t)|dt\right\},\end{aligned}$$

where $\Im m\lambda$ denotes the imaginary part of λ.
Considering two cases $(\Im m\lambda+\eta)\geqq 0$ and $(\Im m\lambda+\eta)<0$, we have

$$(3) \qquad e^{\mu-\eta t_0}|\,f(t_0)\,|\leqq\int_0^{T_0}e^{\mu-\eta t}|\,(D_t-\lambda)f(t)\,|dt.$$

If we write

$$P(D_t, \xi)\tilde{u}(t, \xi)=Q_m(\xi)\prod_{j=1}^{m}(D_t-\lambda_j(\xi))\,\tilde{u}(t, \xi)$$

and set $\mu=a\log\gamma(\xi)$ and $\eta=b\log\gamma(\xi)$, we get (2) by the repeated application of (3).

Lemma 2. *Let $Q(\xi)$ be a differential polynomial (of order $s\geqq 0$) with the principal part $Q^{(0)}(\xi)$, and let Ξ be a bounded domain in R^ν with the diameter $d=d(\Xi)$. Then we have*

$$(4) \qquad \int_{E^\nu}|\,\tilde{v}(\xi)\,|\,d\xi\leqq A_{Q,d}\int_{E^\nu}|\,Q(\xi)\,\tilde{v}(\xi)\,|d\xi,\quad v\in C_0^\infty(\Xi)$$

where $A_{Q,d}=\left(\frac{4d}{\pi}\right)^s(\operatorname{Max}_{|\xi|=1}Q^{(0)}(\xi))^{-1}$.

Proof. After the orthogonal transformation we may assume

$$Q(\xi)=q_s\xi_1^s+\sum_{j=0}^{s-1}q_j(\tilde{\xi})\xi_1^j$$

where q_s is a complex constant such that $|\,q_s\,|=\operatorname{Max}_{|\xi|=1}|\,Q^{(0)}(\xi)\,|$ and $q_j(\tilde{\xi})$ $(0\leqq j\leqq s-1)$ are polynomials in $\tilde{\xi}=(\xi_2, \cdots, \xi_\nu)$. Let $h(x_1)$ be a function of class C_0^∞ in $(r, r+d)$ for some real r. Then, for any complex number τ, we have

$$\int_{E^1}|(\xi_1-\tau)\tilde{h}(\xi_1)|d\xi_1 \geqq R\int_{E^1}|\tilde{h}(\xi_1)|d\xi_1 - R\int_{|\xi_1-\tau|\leqq R}|\tilde{h}(\xi_1)|d\xi_1$$

where

$$\tilde{h}(\xi_1)=\frac{1}{\sqrt{2\pi}}\int_{E^1}e^{-ix_1\xi_1}h(x_1)dx_1.$$

On the other hand

$$|\tilde{h}(\xi_1)|\leqq\frac{1}{\sqrt{2\pi}}\int_{\gamma}^{\gamma+d}|h(x_1)|dx_1\leqq\frac{d}{2\pi}\int_{E^1}|\tilde{h}(\xi_1)|d\xi_1.$$

Hence, setting $R=\pi/(2d)$, we have

$$\int_{E^1}|(\xi_1-\tau)\tilde{h}(\xi_1)|d\xi_1\geqq\frac{\pi}{4d}\int_{E^1}|\tilde{h}(\xi_1)|d\xi_1,$$

so that we have

$$\int_{E^\nu}|Q(\xi)\tilde{v}(\xi)|d\xi=\int_{E^{\nu-1}}\left\{\int_{E^1}|q_s\prod_{j=1}^{s}(\xi_1-\tau_j(\bar{\xi}))\tilde{v}(\xi_1,\bar{\xi})|d\xi_1\right\}d\bar{\xi}$$

$$\geqq|q_s|\left(\frac{\pi}{4d}\right)^s\int_{E^\nu}|\tilde{v}(\xi)|d\xi.$$ Q.E.D.

Lemma 3. *Let Ξ be a bounded domain in R^ν. Then, for $k=-(\nu+1),\cdots,0,1,\cdots$, we have*

(5) $$\int_{E^\nu}(1+|\xi|)^k|\tilde{v}(\xi)|d\xi\leqq A_{\nu,\Xi}2^k\underset{|\beta|\leqq k+\nu+1}{\mathrm{Max}}|D_x^\beta v|,\quad v\in C_0^\infty(\Xi),$$

where $A_{\nu,\Xi}=2(2/\pi)^{\nu/2}$ meas$(\Xi)\int_{E^\nu}(1+|\xi|)^{-(\nu+1)}d\xi$ and meas(Ξ) denotes the measure of Ξ.

Proof. We have

$$\int_{E^\nu}(1+|\xi|)^k|\tilde{v}(\xi)|d\xi\leqq\left(\int_{E^\nu}(1+|\xi|)^{-(\nu+1)}d\xi\right)\sup_{\xi\in E^\nu}(1+|\xi|)^{k+\nu+1}|\tilde{v}(\xi)|,$$

$$(1+|\xi|)^{k+\nu+1}|\tilde{v}(\xi)|\leqq2^{k+\nu+1}\underset{|\beta|\leqq k+\nu+1}{\mathrm{Max}}|\xi^\beta\tilde{v}(\xi)|$$

and

$$|\xi^\beta\tilde{v}(\xi)|\leqq\frac{1}{\sqrt{2\pi}^{\,\nu}}\int_{E^\nu}|D_x^\beta v(x)|dx.$$

Hence, we get easily (5). Q.E.D.

§ 2. Propagation of regularity. Let Ξ be a bounded domain in R^ν and set $\Omega_{T_0}=(0,T_0)\times\Xi$ $(T_0>0)$.

Theorem 1. *Let $u(t,x)$ be a classical solution of $P(D_t,D_x)u(t,x)=f(t,x)$ for $f\in C(\Omega_{T_0})$.*

Assume that f is infinitely differentiable in x for any fixed $t\in(0,T_0)$ and the mapping

(6) $$f:(0,T_0)\ni t\to f(t,\cdot)\in\mathcal{E}(\Xi)$$

is continuous,[2] *furthermore assume that, for some constant $\delta>0$, $D_t^ju(j=0,\cdots,m-1)$ are infinitely differentiable function of x in*

2) We call the mapping $f:(0,T_0)\ni t\to f(t,\cdot)\in\mathcal{E}(\Xi)$ is continuous, if, for any fixed compact set K of Ξ, α and $t_0\in(0,T_0)$, $D_x^\alpha f(t,x)\to D_x^\alpha f(t_0,x)$ as $t\to t_0$ uniformly on K.

$((0, \delta)\times\Xi)\cup((0, T_0)\times\Xi_\delta)$[3] *and the mappings*

$$D_t^j u: \begin{cases} (0, \delta)\ni t\to D_t^j u(t, \cdot)\in\mathcal{E}(\Xi) \\ (0, T_0)\ni t\to D_t^j u(t, \cdot)\in\mathcal{E}(\Xi_\delta) \end{cases} \quad (j=0, 1, \cdots, m-1) \tag{7}$$

are continuous.

Then, $D_t^j u(t, x)$ $(j=0, 1, \cdots, m)$ *are infinitely differentiable functions of* x *in* Ω_{T_0} *and the mappings*

$$D_t^j u: (0, T_0)\ni t\to D_t^j u(t, \cdot)\in\mathcal{E}(\Xi) \qquad (j=0, 1, \cdots, m) \tag{8}$$

are continuous.

Proof. We fix T, T', T'', and δ' such that $0<T<T'<T''<T_0$ and $0<\delta'<\delta$. Take a function $\Psi(t, x)\in C_0^\infty(\Omega_{T_0})$ such that $\Psi\equiv 1$ in $(\delta', T'')\times(\Xi-\Xi_{\delta'})$ where $\Xi-\Xi_{\delta'}=\{x; x\in\Xi, x\notin\Xi_{\delta'}\}$.

Set $U=\Psi u$, then

$$P(D_t, D_x)U=\Psi f+f'\equiv F, \tag{9}$$

where

$$f'=\sum_{j+|\alpha|\neq 0}\frac{1}{j!\,\alpha!}D_t^j D_x^\alpha\Psi\cdot P^{(j,\alpha)}(D_t, D_x)u\left(P^{(j,\alpha)}(\lambda, \xi)=\frac{\partial^{j+|\alpha|}}{\partial\lambda^j\partial\xi^\alpha}P(\lambda, \xi)\right).$$

Since $f'\equiv 0$ in $(\delta', T'')\times(\Xi-\Xi_{\delta'})$, we see, by the assumption of Theorem 1, that $F\in C_0(\Omega_{T_0})$ and a infinitely differentiable function of x in $\Omega_{T''}=(0, T'')\times\Xi$, and that, for any α and $t_0\in(0, T'')$,

$$D_x^\alpha F(t, x)\to D_x^\alpha F(t_0, x) \quad \text{as } t\to t_0 \tag{10}$$

uniformly in Ξ. Set $a=(n+\nu+1)T'/(T'-T)$, $b=(n+\nu+1)/(T'-T)$. Then we have

$$a-bt\leqq a \text{ in } (0, T_0),\ \geqq n \text{ in } (0, T),\ \leqq-(\nu+1) \text{ in } (T', T_0). \tag{11}$$

Approximating U by $U_n\in C_0^\infty(\Omega_{T_0})$ and applying (2) to U_n by setting $\gamma(\xi)=(1+|\xi|)$, we get by (11)

$$\begin{aligned} &\int_{E^\nu}(1+|\xi|)^n|Q_m(\xi)\tilde{U}(t_0, \xi)|d\xi \\ &\leqq T_0^{m-1}\left\{\int_0^{T'}\int_{E^\nu}(1+|\xi|)^a|\tilde{F}(t, \xi)|d\xi dt\right. \\ &\qquad\left.+\int_{T'}^{T_0}\int_{E^\nu}(1+|\xi|)^{-(\nu+1)}|\tilde{F}(t, \xi)|d\xi dt\right\} \end{aligned} \tag{12}$$

for every $t_0\in(0, T)$.

By Lemma 2 and 3 we have for $|\alpha|=n$

$$\begin{aligned} &|D_x^\alpha U(t_0, x)|\leqq\frac{1}{\sqrt{2\pi}^{\,\nu}}\int_{E^\nu}|\widetilde{D_x^\alpha U}(t, \xi)|d\xi \\ &\leqq T_0^{m-1}A_{Q_m,\nu,\Xi}\left\{2^a\int_0^{T'}\underset{\substack{|\beta|\leqq a+\nu+1\\ x\in\Xi}}{\mathrm{Max}}|D_x^\beta F(t, x)|dt+2^{-(\nu+1)}\int_{T'}^{T_0}\underset{x\in\Xi}{\mathrm{Max}}|F(t, x)|dt\right\} \end{aligned} \tag{13}$$

for $t_0\in(0, T)$. Since we can take n arvitrarily large, we get, in $(\delta', T)\times(\Xi-\Xi_{\delta'})$, $u(t_0, x)=U(t_0, x)$ is a infinitely differentiable function of x. Letting $T\to T_0$, we get by (7) that $u(t, x)$ is a infinitely

3) $\Xi_\delta=\{x\in\Xi;\ \mathrm{dis}(x, \partial\Xi)<\delta\}$ where $\mathrm{dis}(x, \partial\Xi)$ means the distance from x to the boundary $\partial\Xi$ of Ξ.

differentiable function of x in

$$\Omega_{T_0}=((0,\delta)\times\Xi)\cup((0,T_0)\times\Xi_\delta)\cup((0,T_0)\times(\Xi-\Xi_{\delta'})).$$

In order to prove the continuity of the mappings (8), we use (13) by replacing $U(t)$ by $(U(t+h)-U(t))$. Then $P(U(t+h)-U(t))=(F(t+h)-F(t))$. By (10) we see that (13) has meaning for $h<T''-T'$, so that we have

$$D_x^\alpha u(t_0+h)\to D_x^\alpha u(t_0) \quad \text{as} \quad h\to 0$$

uniformly in $(\delta', T)\times(\Xi-\Xi_{\delta'})$ for any fixed α. Hence, letting $T\to T_0$ we get the continuity of the mapping $u\colon (0,T_0)\ni t\to u(t,\cdot)\in\mathcal{E}(\Xi)$. Next, setting $u_1=D_tu$, we have $P_1(D_t,D_x)u_1\equiv\sum_{j=1}^m Q_j(D_x)D_t^{j-1}u_1=(f-Q_0(D_x)u)\equiv f_1$. Then u_1 and f_1 satisfy the conditions of Theorem 1, so that the mapping

$$D_tu=u_1\colon (0,T_0)\ni t\to u_1(t,\cdot)\in\mathcal{E}(\Xi)$$

is continuous, and so on we get the continuity of the mappings

$$D_t^ju\colon (0,T_0)\ni t\to D_t^ju(t,\cdot)\in\mathcal{E}(\Xi) \qquad (j=2,\cdots,m-1).$$

Finally we write $Q_m(D_x)D_t^mu=f-\sum_{j=0}^{m-1}Q_j(D_x)D_t^ju$, and by using Lemma 2 and 3 we get the continuity of the mapping

$$D_t^mu\colon (0,T_0)\ni t\to D_t^mu(t,\cdot)\in\mathcal{E}(\Xi).$$

This completes the proof. Q.E.D.

Corollary. *Let $u(t,x)$ be a classical solutions of $P(D_t,D_x)u(t,x)=f(t,x)$ in Ω_{T_0}. Assume that $f\in C^\infty(\Omega_{T_0})$ and that, for some constant $\delta>0$, $u\in C^\infty$ in $((0,\delta)\times\Xi)\cup((0,T_0)\times\Xi_\delta)$. Then, we have $u\in C^\infty(\Omega_{T_0})$.*

Proof. It is easy to see that f and u satisfy the conditions of Theorem 1, so that the mappings

$$(14) \qquad D_t^ju\colon (0,T)\ni t\to D_t^ju(t,\cdot)\in\mathcal{E}(\Xi) \qquad (j=0,1,\cdots,m)$$

are continuous. Setting $u_m=D_t^mu$, we can write $Q_m(D_x)u_m=f-\sum_{j=0}^{m-1}Q_j(D_x)D_t^ju\equiv F$ and for any β

$$D_x^\beta Q(D_x)(u_m(t+h)-u_m(t))/h=i\int_0^1 D_x^\beta D_tF(t+\theta h,x)d\theta.$$

Hence by Lemma 2 and 3 we get the existence of $D_t^{m+1}D_x^\alpha u=D_x^\alpha D_tu_m$ in Ω_{T_0}, and the continuity of the mapping

$$D_t^{m+1}u\colon (0,T_0)\ni t\to D^{m+1}u(t,\cdot)\in\mathcal{E}(\Xi).$$

Writing $Q_m(D_x)D_t^{l+m}u=D_t^lf-\sum_{j=0}^{m-1}Q_j(D_x)D_t^{l+j}u$, we get $u\in C^\infty(\Omega_{T_0})$ by repeated applications of the above discussion for $j=1,2,\cdots$. Q.E.D.

About the propagation of analyticity, using the method of J. Boman [1] and playing the same discussion as the proof of Theorem 1, we get the following without the proof.

Theorem 2. *Let $u(t,x)$ be a classical solution of $P(D_t,D_x)u(t,x)=f(t,x)$ in Ω_{T_0}. Assume f and u satisfy the conditions of Theorem 1, and furthermore we assume that, for any T $(0<T<T_0)$, there exist constants M_T and C_T such that*

$$|D_x^\alpha f| \leqq M_T C_T^{|\alpha|} |\alpha|^{|\alpha|} \quad \text{in } \Omega_T = (0, T) \times \Xi,$$
$$|D_t^j u| \leqq M_T C_T^{|\alpha|} |\alpha|^{|\alpha|} \quad \text{in } (0, T) \times \Xi_\delta \qquad (j = 0, 1, \cdots, m-1),$$

and

$$|D_t^j u| \leqq M C^{|\alpha|} |\alpha|^{|\alpha|} \quad \text{in } (0, \delta) \times \Xi \qquad (j = 0, 1, \cdots, m-1)$$

for some constants M, C.

Then, for any T $(0 < T < T_0)$, *there exist constants* M'_T *and* C'_T *such that*

$$|D_t^j D_x^\alpha u| \leqq M'_T C'^{|\alpha|}_T |\alpha|^{|\alpha|} \text{ in } (0, T) \times \Xi \qquad (j = 0, 1, \cdots, m).$$

Corollary. *Let* $u(t, x)$ *be a classical solution of* $P(D_t, D_x)u(t, x) = f(t, x)$ *in* Ω_{T_0}. *Assume that* f *is analytic in* Ω_{T_0} *and that, for some constant* $\delta > 0$, u *is analytic in* $((0, \delta) \times \Xi) \cup ((0, T_0) \times \Xi_\delta)$. *Then,* u *is analytic in* Ω_{T_0}.

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