82. On Axiom Systems of Propositional Calculi. XIX

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In previous notes (see [1], [2], and [5]), we have proved that equivalential calculus is characterized by each of the axiom systems as follows:

1 *EEpqEqp*, *EEEpqrEpEqr*,

2 *EEpqEEprErq*,

where the fundamental axiom system of equivalential calculus is given by

3 *EEEprEqpErq*, *EEpEqrEEpqr*,

and *E* is the truth functor in the calculus (see [4]).

First our attempt in this paper is to give a proof of the following

Theorem 1. *The equivalential calculus is characterized by each of the following single axiom systems*:

4 *EEpqEErqEpr*,

5′ *EEpqEErpEqr*,

6′ *EEEpqrEqErp*,

7′ *ErEEqErpEpq*.

Proof. The proofs will be carried out by using the two rules of inference, i.e. the rule of substitution and the rule of detatchment. We shall also use prooflines by J. Lukasiewicz for the proof of theses.

4 *p*/*Epq*, *q*/*EErqEpr*, *r*/*Epq* **C*4—*C*4—5,

5 *EEpqEpq*.

4 *p*/*Epq*, *q*/*Epq* **C*5—6,

6 *EErEpqEEpqr*.

6 *p*/*Erq*, *q*/*Epr*, *r*/*Epq* **C*4—7,

7 *EEErqEprEpq*.

7 *q*/*p*, *r*/*p* **C*5 *q*/*p*—8,

8 *Epp*.

4 *p*/*q*, *r*/*p* **C*8 *p*/*q*—9,

9 *EEpqEqp*.

4 *p*/*EEprEqp*, *q*/*Eqr*, *r*/*Erq* **C*7 *p*/*q*, *q*/*r*, *r*/*p*—*C*9 *p*/*r*—10,

10 *EEEprEqpErq*.

9 *p*/*EErpEqr*, *q*/*Epq* **C*10 *p*/*r*, *r*/*p*—11,

11 *EEpqEErpEqr*.

4 p/Epq, $q/EErpEqr$, $r/EEqrErp$ *$C11$—$C9$ p/Eqr,
q/Erp—12,

12 $EEpqEEqrErp$.

11 p/Epq, $q/EEqrErp$, r/Eqp *$C12$—$C9$ p/q,
q/p—13,

13 $EEEqrErpEqp$.

9 $p/EEprErq$, q/Epq *$C13$ p/q, q/p—14,

14 $EEpqEEprErq$.

Thesis 14 is a single axiom 2 of equivalential calculus given above.

Next we shall prove from the adopted axiom:

5′ $EEpqEErpEqr$,

the following theses:

5′ p/Epq, $q/EErpEqr$, r/s *$C5'$—15,

15 $EEsEpqEEErpEqrs$.

15 p/Erp, q/Eqr, r/s, s/Epq *$C5'$—16,

16 $EEEsErpEEqrsEpq$.

16 $q/EqEpq$, r/Epq *$C15$ p/Epq, q/p, r/q—17,

17 $EpEqEpq$.

15 p/r, q/Eqr, r/p, s/q *$C17$ p/q, q/r—18,

18 $EEEprEEqrpq$.

18 q/Epp, r/Epp *$C15$ q/p, r/p, s/p—19,

19 Epp.

5′ p/q, r/p *$C19$ p/q—20,

20 $EEpqEqp$.

5′ p/Eqp, $q/EErqEpr$, r/Epq *$C5'$ p/q, q/p—
$C20$—21,

21 $EEErqEprEpq$.

20 $p/EErqEpr$, q/Epq *$C21$—22,

22 $EEpqEErqEpr$.

Thesis 22 is a new single axiom 4 of equivalential calculus proved above.

C. A. Meredith and A. N. Prior proved that our adopted axiom 6′ is equivalent to $EEpqEErpEqr$ (see [3]).

Lastly we shall prove that the adopted axiom 7′ is a single axiom of equivalential calculus.

7′ p/t, q/s, $r/ErEEqErpEpq$ *$C7'$—23,

23 $EEsEErEEqErpEpqtEts$.

23 $s/EqErp$, $t/EEpqr$ *$C7'$ p/Epq, q/r, $r/EqErp$—24,

24 $EEEpqrEqErp$.

Thesis 24 is a new single axiom 6′. Now we have completed the proof of Theorem 1.

Next we shall prove the following

Theorem 2. *The equivalential calculus is characterized by*

25 *EpEqEqp*,

26 *EEpEqrEEpqr*.

Proof. In the equivalential calculus, we have the following axiom system:

27 *EEpqEqp*,

28 *EEEpqrEpEqr*.

Hence,

28 *r*/*Eqp* **C*27—29,

29 *EpEqEqp*.

Conversely, assume theses 25 and 26, then

26 *r*/*Eqp* **C*25—30,

30 *EEpqEqp*,

which completes the proof of Theorem 2.

Therefore, the following theses are equivalent:

1) *EEEprEEqpErq*, *EEpEqrEEpqr*,
2) *EEpqEqp*, *EEEpqrEpEqr*,
3) *EEpqEEprErq*,
4) *EEpqEErqEpr*,
5) *EEpqEErpEqr*,
6) *EEEpqrEqErp*,
7) *ErEEqErpEpq*,
8) *EpEqEqp*, *EEpEqrEEpqr*.

References

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