78. A Remark on a Theorem of H. Araki

By Katuyosi MATOBA
Kitano Senior High School

(Comm. by Kinjirô KUNUGI, M.J.A., April 12, 1966)

1. On the symposium on operator algebras and its applications in physics held at Research Institute for Mathematical Sciences of Kyoto University in the last December, Prof. H. Araki announced, among many others, the following theorem:

Theorem 1. *Let $\mathfrak{A}_i$ be von Neumann algebras acting on a Hilbert space $\mathfrak{H}$ for $i=1, 2, \cdots$. If $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ is a factorization and $A_i \in \mathfrak{A}_i$, $\|A_i\| \leqq 1$ for evey i, then*

$$\underset{i\to\infty}{\text{w-lim}}\ [A_i-(A_ix \mid x)]=0,$$

for an arbitrary $x \in \mathfrak{H}$ with $\|x\|=1$, where w-lim *means the weak operator limit.*

The theorem is a special case of Prof. Araki's more general theorems, cf. [1; § 2, Prop. 4] and [2; § 15, Cor. 2 to Theorem in Remark 5]. However, in the present note, we shall discuss Theorem 1 in the case of the infinite direct product of von Neumann algebras. Since our proof is quite elementary, it may be observed with some interest. Besides, we shall give a sufficient condition that the convergence becomes strong one. Finally, we shall present an example which shows that our theorem fails when we take the complete infinite direct product.

We should like to express our hearty thanks to Prof. H. Araki for his kind guidance.

2. Let $\mathfrak{H}$ be an incomplete infinite direct product of Hilbert spaces $\mathfrak{H}_1, \mathfrak{H}_2, \cdots$ in the sense of von Neumann [3] and $\mathfrak{K}$ the set of vectors in $\mathfrak{H}$ such that $x=x_1\otimes x_2\otimes \cdots$ and $\sum_i | \|x_i\| -1 | < +\infty$ $(x_i \in \mathfrak{H}_i$, $i=1, 2, \cdots)$. $\mathfrak{K}$ is a total set in $\mathfrak{H}$. Let A_i be an operator acting on the Hilbert space $\mathfrak{H}_i$ for each $i=1, 2, \cdots$. According to [3; Lemma 6.2.4], each A_i can be considered as an operator acting on $\mathfrak{H}$; especially,

$$A_ix=x_1\otimes x_2\otimes \cdots \otimes x_{i-1}\otimes A_ix_i\otimes x_{i+1}\otimes \cdots$$

for $x=x_1\otimes x_2\otimes \cdots \in \mathfrak{K}$.

Under these circumstances, we shall prove the following

Theorem 2. *If A_i is an operator on the Hilbert space $\mathfrak{H}_i$ with $\|A_i\| \leqq 1$ for each i, then*

$$\underset{i\to\infty}{\text{w-lim}}\ [A_i-(A_ix \mid x)]=0, \tag{1}$$

for an arbitrary $x \in \mathfrak{H}$ with $\|x\|=1$.

Proof. Let x, y, and z be vectors in $\mathfrak{K}$ such as $\|x\|=1$,

$$y=y_1\otimes y_2\otimes\cdots \quad \text{and} \quad z=z_1\otimes z_2\otimes\cdots$$

with $y_i, z_i\in\mathfrak{H}_i$ for every i. Then we have $\|y_i-z_i\|\to 0$ $(i\to\infty)$ by [3; Lemma 3.3.4].

By a direct calculation, we have

$$\begin{aligned}&(A_iy\,|\,z)-(A_ix\,|\,x)(y\,|\,z)\\&=\frac{(A_iy_i\,|\,z_i)}{(y_i\,|\,z_i)}(y\,|\,z)-\frac{(A_ix_i\,|\,x_i)}{(x_i\,|\,x_i)}(y\,|\,z)\\&=\{[(A_iy_i\,|\,z_i)-(A_iy_i\,|\,y_i)](x_i\,|\,x_i)+(A_iy_i\,|\,y_i)[(x_i\,|\,x_i)-(y_i\,|\,z_i)]\\&\quad+[(A_iy_i\,|\,y_i)-(A_ix_i\,|\,x_i)](y_i\,|\,z_i)\}(y\,|\,z)/(y_i\,|\,z_i)(x_i\,|\,x_i).\end{aligned}$$

While, we see

$$\begin{aligned}&|(A_iy_i\,|\,z_i)-(A_iy_i\,|\,y_i)|\leqq\|A_iy_i\|\cdot\|y_i-z_i\|\to 0,\\&(x_i\,|\,x_i)-(y_i\,|\,z_i)\to 1-1=0,\end{aligned}$$

and

$$\begin{aligned}|(A_iy_i\,|\,y_i)-(A_ix_i\,|\,x_i)|&\leqq|(A_i(y_i-x_i)\,|\,y_i)|+|(A_ix_i\,|\,y_i-x_i)|\\&\leqq\|y_i-x_i\|(\|y_i\|+\|x_i\|)\to 0,\end{aligned}$$

as $i\to\infty$. Therefore, we have

$$\lim_{i\to\infty}[(A_iy\,|\,z)-(A_ix\,|\,x)(y\,|\,z)]=0, \tag{2}$$

for all $y,z\in\mathfrak{K}$. By $\|A_i-(A_ix\,|\,x)\|\leqq 2$, we can easily deduce that the formula (2) holds for all $y,z\in\mathfrak{H}$.

The restriction that $x\in\mathfrak{H}$ can be relaxed by the fact that

$$\lim_{i\to\infty}[(A_ix\,|\,x)-(A_ix'\,|\,x')]=0$$

for an arbitrary $x'\in\mathfrak{H}$ with $\|x'\|=1$, which follows from (2).

3. In the present section, we shall consider the strong convergence of $\{A_i\}$ under an additional assumption.

Let $\mathfrak{A}_i$ be a von Neumann algebra acting on $\mathfrak{H}_i$ for each i. According to [3; Lemma 6.2.4], each $\mathfrak{A}_i$ can be considered as an algebra acting on $\mathfrak{H}$. We shall denote $\otimes_i\mathfrak{A}_i$ the von Neumann algebra generated by $\mathfrak{A}_1,\mathfrak{A}_2,\cdots$ which is called the *incomplete infinite direct product* of $\mathfrak{A}_1,\mathfrak{A}_2,\cdots$, cf. [4] and [5].

Theorem 3. *Let $A_i\in\mathfrak{A}_i$ such as $\|A_i\|\leqq 1$ for each i. If $A=\otimes_iA_i$ is a non-zero member of $\otimes_i\mathfrak{A}_i$. Then*

$$\operatorname*{s-lim}_{i\to\infty}[A_i-(A_ix\,|\,x)]=0, \tag{3}$$

for any $x\in\mathfrak{H}$ with $\|x\|=1$, where s-lim *means the strong operator limit.*

The proof of the theorem will be divided into two lemmas.

Lemma 1. *Let $A_i\in\mathfrak{A}_i$ such as $\|A_i\|\leqq 1$. If $A=\otimes_iA_i\neq 0$ belongs to $\otimes_i\mathfrak{A}_i$, then A_i converges strongly to* 1.

Proof. Let $\mathfrak{K}$ be as in § 2. Then there is a vector $y\in\mathfrak{K}$ such that $\|y\|=1$ and $Ay\neq 0$. Put $y=y_1\otimes y_2\otimes\cdots$. Then we have for any $x=x_1\otimes x_2\otimes\cdots$ with $\|x\|=1$

$$|(A_i x_i \mid x_i)-1| \leqq |(A_i x_i - A_i y_i \mid x_i)| + |(A_i y_i \mid x_i)-1|$$

and

$$|\,\|A_i x_i\|-1| \leqq \|A_i x_i - A_i y_i\| + |\,\|A_i y_i\|-1|.$$

Since both y and Ay are in $\mathfrak{K}$, we have by [3; Lemma 3.3.4]

$$|(A_i x_i \mid x_i)-1| \to 0 \quad \text{and} \quad |\,\|A_i x_i\|-1| \to 0,$$

as $i \to \infty$. Hence $\|A_i x - x\| \to 0$ for any x in $\mathfrak{H}$, which proves the lemma.

Lemma 2. *If A_i converges strongly to* 1, *then* (3) *holds.*

This lemma follows from the fact that

$$\|[A_i-(A_i x \mid x)]y\|^2 = \|A_i y\|^2 - (A_i x \mid x)(A_i y \mid y)$$
$$-(A_i x \mid x)^*(y \mid A_i y) + |(A_i x \mid x)|^2$$

converges to zero as $i \to \infty$.

4. At this end, we wish to present an example that Theorem 2 fails when $\mathfrak{H}$ is the *complete* infinite direct product of $\mathfrak{H}_1, \mathfrak{H}_2, \cdots$

Let $\mathfrak{H}_i$ be the two-dimensional Hilbert space and

$$A_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ for each } i=1, 2, \cdots$$

If

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots$$

and

$$y = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \cdots,$$

then $(A_i y \mid y) - (A_i x \mid x)(y \mid y) = 1$ for all $i = 1, 2, \cdots$. Hence (1) is impossible for the complete infinite direct product.

References

[1] H. Araki: On the algebra of all local observables. Progress of Theo. Phys., **32**, 844-854 (1964).

[2] ——: Local Quantum Theory (to be published by Benjamin).

[3] J. von Neumann: On infinite direct products, Coll. Works, Vol. 3. Pergamon, New York, 323-399 (1961).

[4] T. Saito: On incomplete infinite direct product of W*-algebras. Tôhoku Math. J., **10**, 165-171 (1958).

[5] Z. Takeda: Inductive limit and infinite direct product of operator algebras. Tôhoku Math. J., **7**, 67-86 (1955).