76. *Counter Examples in the Theory of Generalized Uniform Spaces*

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(Comm. by Kinjirô KUNUGI, M.J.A., April 12, 1966)

Consider a class of topological spaces R in which the neighborhood of the points of R can be determined by a set of index α so that we can denote it by $V_\alpha(p)$.

We shall call them generalized uniform spaces. We shall in this note discuss five axioms concerning ganeralized uniform spaces as follows:

Let us suppose that, for any point p of R and for any index α, an index $\beta=(p,\alpha)$ is so fixed, that one of the five following postulates can be satisfied.*)

As to any point q,

(A) in the case of $\{V_\beta(p)\cdot V_\beta(q)\neq 0\}$, we have
$$\{V_\beta(q)\subset V_\alpha(p)\}.$$
(B) in the case of $\{V_\beta(p)\cdot V_\beta(q)\neq 0\}$, we have
$$\{q\in V_\alpha(p)\}.$$
(C) in the case of $\{q\in V_\beta(p)\}$, we have
$$\{V_\beta(q)\subset V_\alpha(p)\}.$$
(D) in the case of $\{p\in V_\beta(q)\}$, we have
$$\{V_\beta(q)\subset V_\alpha(p)\}.$$
(E) in the case of $\{p\in V_\beta(q)\}$, we have
$$\{q\in V_\alpha(p)\}.$$

We have evidently the following relations

(1) (A)$\Rightarrow$(B),
(2) (B)$\Rightarrow$(E),
(3) (A)$\Rightarrow$(C),
(4) (A)$\Rightarrow$(D),
(5) (D)$\Rightarrow$(E);

Now we can ask whether their converses are true. In this note we shall prove none of them are true.

Example 1. An example by which we can see that the converse (1) is not true.

Let S be a straight line of real number. For any point p of S and any positive number α, define $V_\alpha(p)$ as follows, where $V_\alpha(p)$ is a neighborhood of p. If $p>0$, there $V_\alpha(p)$ is an open interval

*) c.f. Takeshi Inagaki: Ten Shugoron. Iwanami (Japanese).

determined by $p-\alpha<x<p+\alpha$. If $p=0$, then $V_\alpha(0)$ is a half open interval $0\leq x<\alpha$. If $p<0$ and $|p|\geq\alpha$, then $V_\alpha(p)$ is an open interval $p-\alpha<x<p+\alpha$. If $p<0$ and $|p|<\alpha$, then $V_\alpha(p)$ is an open interval $p-|p|<x<p+|p|=0$. Let R be a space obtained from S modifying its topology in the above way.

(B) is true in the space R for the following reason.

If $p>0$, then we take $\beta=\beta(p,\alpha)=\frac{\alpha}{2}$; if $V_\beta(p)\cdot V_\beta(q)\neq 0$ then $q\in V_\alpha(p)$ for any point q.

If $p=0$, then we take $\beta=\beta(0,\alpha)=\frac{\alpha}{2}$; if $V_\beta(0)\cdot V_\beta(q)\neq 0$ then $q\in V_\alpha(0)$ for any point q.

If $p<0$, then we take $\beta=\beta(p,\alpha)=\min\left(\frac{\alpha}{2},\frac{|p|}{2}\right)$; if $V_\beta(p)\cdot V_\beta(q)\neq 0$, then $q\in V_\alpha(p)$ for any point q.

(A) is not true in the space R for the following reason.

For point 0, whatever index $\beta=\beta(0,\alpha)$ we may take, we have $V_\beta(0)\cdot V_\beta(q)\neq 0$ where q is a point which satisfies $0<q<\beta$. And also we have $V_\beta(q)\not\subset V_\alpha(0)$, because the set $V_\beta(q)=\{x; q-\beta<x<q+\beta\}$ contains negative points, whereas $V_\alpha(0)$ consists only of positive points and 0.

Example 2. An example which shows that the converse (2) is not true.

Let S is closed interval $[0, 1]$ and its topology is as follows.

Let D be a set of all numbers $\frac{1}{n}$, $n=1, 2, \cdots$. Define $J(p)$ as follows. If p is $0<p<1$, then $J(p)$ is an open interval containing p. If $p=0$ or $P=1$, then $J(p)$ is a half open interval $0\leq x<a$, or $a<x\leq 1$, where a is any point between 0 and 1. If $p\neq 0$ and $p\notin D$, then every $J(p)$ is the neighborhood of p; if $p=0$, then the set $J(0)-D$ is a neighborhood of 0. If $p\in D$, then the set of form $J(p)-0$ is the neighborhood of p. $V_\alpha(p)$ means that $J(p)$ is the set of $p-\alpha<x<p+\alpha$.

(E) is true in the space S for the following reason.

When we take $\beta=\beta(p,\alpha)=\alpha$, for any point p; if $p\in V_\beta(q)$ then $q\in V_\alpha(p)$ for any point q.

(B) is not true in the space S for the following reason.

Whatever index $\beta=\beta(0,\alpha)$, we may choose one of the neighborhoods of the point 0, say $V_\beta(0)$, such that $\min(\alpha, 2\beta)>\frac{1}{n}$ implies $V_\beta(0)\cdot V_\beta\left(\frac{1}{n}\right)\neq 0$ and $\frac{1}{n}\notin V_\alpha(0)$.

Example 3. An example by which we can see that the converse

(3) is not true.

Let X be the closed interval $[0, 1]=[0\leq x\leq 1]$ and Y be the closed interval $[0, 1]=[0\leq y\leq 1]$.

Consider product space $T=X\times Y$, and modify the neighborhood of the space T.

Let (x, y) be a point of $T=X\times Y$ and let $\varepsilon, \varepsilon'$ be any positive number.

The neighborhood of point (x, y) will be a set of any point (x', y') which will satisfy such inequalities as $x-\varepsilon<x'\leq x$ and $y-\varepsilon'<y'\leq y$.

(c) is true in the space T, for the following reason. When we take the index $\beta=\beta(p, \alpha)=\frac{\alpha}{2}$ for any point $p(x, y)$, we have $V_\beta(q)\subset V_\alpha(p)$ for any point q which will satisfy $q\in V_\beta(p)$. But,

(A) is not true in thes pace T, for the following reason. Whatever index β we may take for any point p on the line $y=1-x$, $V_\beta(p)\cdot V_\beta(q)\neq 0$ is true by point q which is, for instance, on the line $y=1-x$ and whose distance from point p is shorter than β.

And $V_\beta(q)\not\subset V_\alpha(p)$, because of $q\notin V_\alpha(p)$.

Example 4. An example which shows that the converse (4) is not true.

Space S is the part which will satisfy $y\geq 0$ on the XY-plane. The neighborhood $V_\alpha(p)$ of a point $P(x, y)$ on the space S is,

(i) In the case of $y>0$; the interior of a circle with radius α and center p and whose part is $y>0$.

(ii) In the case of $y=0$; point p and the interior of a circle with radius α which contracts at point p and the X-axis and whose part is $y>0$. (D) is true in the space S for the following reason.

When we take the index $\beta=\beta(p, \alpha)<\min\left(\frac{\alpha}{2}, \frac{y}{2}\right)$ for any point $p(x, y)\ y\neq 0$, we have $V_\beta(q)\subset V_\alpha(p)$ for any point q which will satisfy $p\in V_\beta(q)$.

In the case of $p(x, 0)$, $p\in V_\beta(q)$ means $p=q$, therefore take $\beta=\beta(p, \alpha)=\alpha$, then $V_\beta(q)\subset V_\alpha(p)$.

(A) is not true in the space S, for the following reason.

Whatever index $\beta=\beta(p, \alpha)$ we may take by any point p on X-axis, $V_\beta(p)\cdot V_\beta(q)\neq 0$ is true by point q which is on X-axis and whose distance from point p is shorter than β. And $V_\beta(q)\not\subset V_\alpha(p)$ because of $q\notin V_\alpha(p)$.

Example 5. An example by which we can see that the converse (5) is not true.

Consider the space S of example 2.

(E) is true in the space S of example 2.

(D) is not true in the space S (of example 2), for the following reason.

Whatever index $\beta=\beta(0, \alpha)$ we may take for point 0, we have $0 \in V_\beta(q)$ for point q whose distance from point 0 is shorter than β and which will satisfy $q \notin D$.

And also we have $V_\beta(q) \not\subset V_\alpha(0)$, because point of set D is not contained within the neighborhood $V_\alpha(0)$.