107. Γ-Bundles and Almost Γ-Structures

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In [2]-II, the author states that if X is a normal paracompact topological space, then we can define a sheaf of groups $H_*(n)_c$ over X and there is a 1 to 1 correspondence between the set of equivalence classes of n-dimensional topological microbundles over X and $H^1(X, H_*(n)_c)$. In this note, first we give the precise definition of $H_*(n)_c$ and (topological) connection of topological microbundles. Next, using $H_*(n)_c$, we define the almost Γ-structure if X is a topological manifold and give an integrability condition of almost Γ-structures.

1. *Definition of the sheaf* $H_*(n)_c$. We denote the semigroup of all homeomorphisms of $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$ which fix the origin by $E_0(n)$. $E_0(n)$ is regarded to be a topological semigroup by compact open topology. We denote by X a topological space with $\{U_\alpha(x)\}$ the neighborhood basis of x. The semigroup of all continuous maps from $U_\alpha(x)$ into $E_0(n)$ is denoted by $H(U_\alpha(x), E_0(n))$. For $f \in H(U_\alpha(x), E_0(n))$, we set

$$f(y, a) = (y, f(y)(a)).$$

By definition, f is a homeomorphism from $U_\alpha(x) \times \boldsymbol{R}^n$ into $U_\alpha(x) \times \boldsymbol{R}^n$.

Definition. We call f and g are equivalent if f and g coincide on some neighborhood of $x \times 0$ in $U_\alpha(x) \times \boldsymbol{R}^n$.

The set of equivalence classes of $H(U_\alpha(x), E_0(n))$ by this relation is denoted by $H_*(U_\alpha(x), E_0(n))$.

If $U_\alpha(x)$ contains $U_\beta(x)$, then there is a homeomorphism $\bar{r}^\alpha_\beta$: $H_*(U_\alpha(x), E_0(n)) \to H_*(U_\beta(x), E_0(n))$ induced from the restriction homeomorphism. We set

$$(1) \qquad H_*(n)_x = \lim [H_*(U_\alpha(x), E_0(n)), \bar{r}^\alpha_\beta].$$

Lemma 1. *$H_*(n)_x$ is a group.*

If $f \in H(U, E_0(n))$, then its class in $H_*(n)_x$ is denoted by f_x. We set

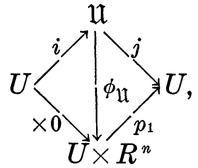
(2) $U(f_x, V(x)) = \{f_y \mid y \in V(x)\}$, $V(x)$ is a neighborhood of x in X.

In $\bigcup_{x \in X} H_*(n)_x$, we take $\{U(f_x, V(x))\}$ to be the neighborhood basis of f_x, then $\bigcup_{x \in X} H_*(n)_x$ becomes a sheaf of groups over X. We denote this sheaf by $H_*(n)_c$.

2. *The cohomology set* $H^1(X, H_*(n)_c)$. **Theorem 1.** *If X is a normal paracompact topological space, then there is a* 1 *to* 1

correspondence between the set of all equivalence classes of n-dimensional topological microbundles over X and $H^1(X, H_*(n)_c)$.

Proof. If $\mathfrak{X}: X \xrightarrow{i} E \xrightarrow{j} X$ is an n-dimensional microbundle defined by



then setting

$$\phi_{\mathfrak{U}}\phi_{\mathfrak{V}}^{-1}(x, a)=(x, \bar{\psi}_{UV}(x)(a)),\ (x, a)\in\phi_{\mathfrak{V}}(\mathfrak{U}\cap\mathfrak{V}),$$

$\{\bar{\psi}_{UV}(x)\}$ induces an element $k(\mathfrak{X})$ of $H^1(X, H_*(n)_c)$ and it is determined by the equivalence class of $\mathfrak{X}$.

On the other hand, for $\{\psi_{\alpha\beta}(x)\}\in H^1(X, H_*(n)_c)$, we take the representation $\bar{\psi}_{\alpha\beta}(x)$ of $\psi_{\alpha\beta}(x)$ and assume $\{U_\alpha\}$ is locally finite. Then for some neighborhood of the origin $Q_{\alpha\beta}$, we get

$$\bar{\psi}_{\beta\gamma}(x)\bar{\psi}_{\gamma\alpha}\bar{\psi}_{\alpha\beta}(x)(a)=a,\ \text{if}\ x\in U_\alpha\cap U_\beta\cap U_\gamma,\ a\in Q_{\alpha\beta}.$$

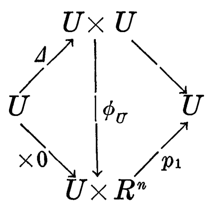
In $U_\alpha\times \boldsymbol{R}^n\times\alpha$, we set

$$\mathfrak{U}_{\beta\alpha}=((U_\alpha\cap U_\beta)\times Q_{\beta\alpha}\cap\hat{\bar{\psi}}_{\alpha\beta}((U_\alpha\cap U_\beta)\times Q_{\alpha\beta}))\times\alpha.$$

By lemma 1, $\mathfrak{U}_{\beta\alpha}$ is an open set of $U_\alpha\times\boldsymbol{R}^n\times\alpha$. We identify $\mathfrak{U}_{\alpha\beta}\ni x\times a\times\beta$ and $x\times\bar{\psi}_{\alpha\beta}(x)(a)\times\alpha\in\mathfrak{U}_{\beta\alpha}$. Then setting E the quotient space of $\bigcup_\alpha U_\alpha\times\boldsymbol{R}^n\times\alpha$ by this relation, we can define $i: X\to E$ and $j: E\to X$, and $\mathfrak{X}: X\xrightarrow{i}E\xrightarrow{j}X$ is a topological microbundle with $k(\mathfrak{X})=\{\psi_{\alpha\beta}(x)\}$.

Since we know if $\{\bar{\psi}_{\alpha\beta}'(x)\}$ is another representation of the class of $\{\psi_{\alpha\beta}(x)\}\in H^1(X, H_*(n)_c)$, and $\mathfrak{X}'$ is constructed from $\{\bar{\psi}_{\alpha\beta}'(x)\}$ then $\mathfrak{X}$ and $\mathfrak{X}'$ are equivalent, we obtain the theorem.

Example. If X is a topological manifold with coordinate nighborhood system $\{(U, h_U)\}$, then the tangent microbundle $\tau: X\xrightarrow{\Delta}X\times X\xrightarrow{p}X$ is defined by the diagram



$$\phi_U(x, y)=(x, h_U(y)-h_U(x)),$$

where Δ is the diagonal map. ([5]). Therefore, setting

$$h_{U,x}(y)=h_U(y)-h_U(x),\ y\in U,$$

the transition functions $\{g_{UV}(x)\}$ of τ is given by

$$(3)\qquad g_{UV}(x)(a)=h_{U,x}h_{V,x}^{-1}(a)=h_Uh_V^{-1}(a+h_V(x))-h_U(x).$$

We set $g_{UV}=h_Uh_V^{-1}$ and

$$(4)\qquad t_{U,x}(a)=a-h_U(x),$$

then we have

(3)′ $$g_{UV}(x)=t_{U,x}g_{UV}t_{V,x}^{-1}.$$

3. *Connection of topological microbundles.* Since the sheaf $H_*(n)_c$ is defined on $\overbrace{X\times\cdots\times X}^{r}$, we can define $C^r(X, H_*(n)_c)$ similarly as $C^r(X, G)$. (Cf. [2]-I).

Definition. If the collection $\{s_U\}$, $s_U\in C^1(U, H_*(n)_c)$, satisfies

(5) $$\varphi_{UV}(x_0)^{-1}s_U(x_0, x_1)\varphi_{UV}(x_1)=s_V(x_0, x_1),$$

for $\{\varphi_{UV}(x)\}=k(\mathfrak{X})\in H^1(X, H_*(n)_c)$, then we call $\{s_U\}$ is a connection form of $\mathfrak{X}$.

We note that the results of [2] are also true for this connection form.

Although we do not know the existence of connection forms of topological microbundles or even of tangent microbundles, if X is a topological manifold, then setting $t_{U,x,y}=t_{U,x}t_{U,y}^{-1}$, we obtain

(6) $$g_{UV}(x)^{-1}t_{U,x,y}g_{UV}(y)=t_{V,x,y},$$

by (3)′. We call $\{t_{U,x,y}\}$ the pseudoconnection of X.

4. *Almost Γ-structures.* We denote by Γ the pseudogroup consisted by a class of homeomorphism from some open set of $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$. According to [7], we define

Definition. A topological manifold X is called a Γ-manifold if $g_{UV}(=h_U h_V^{-1})$ belongs in Γ for all $\{U, V\}$.

Example 1. If Γ is the pseudogroup of all orientation preserving maps, then a Γ-manifold is an oriented manifold.

Example 2. If Γ is the pseudogroup of all diffeomorphisms, then a Γ-manifold is a smooth manifold.

Example 3. If $\boldsymbol{R}^n=\boldsymbol{C}^m$ $(n=2m)$ and Γ is the pseudogroup of all holomorphic maps, then a Γ-manifold is a complex manifold.

We assume that Γ containes all parallel transformations of $\boldsymbol{R}^n$ and set

$\Gamma_0=\{f\,|\,f\in\Gamma, f$ is defined on some neighborhood of 0 and $f(0)=0\}$.

Note. This assumption about Γ is satisfied by the pseudogroups of the above examples and five of primitive infinite continuous pseudogroups of Cartan. But one of Cartan's primitive infinite continuous pseudogroup does not satisfy this assumption.

We give the compact open topology for Γ_0, then starting from Γ_0, we can construct a sheaf of groups Γ_{*c} similarly as $H_*(n)_c$.

Lemma 2. *Γ_{*c} is a subsheaf of $H_*(n)_c$.*

We denote the inclusion from Γ_{*c} into $H_*(n)_c$ by $i(=i_\Gamma)$. i induces the map $i^*\colon H^1(X, \Gamma_{*c})\to H^1(X, H_*(n)_c)$.

Lemma 3. *If X is a Γ-manifold, then the tangent microbundle τ of X is in $i_\Gamma{}^*$-image.*

Definition. A topological manifold X is called an almost

Γ-manifold if the tangent microbundle τ of X belongs in $i_\Gamma{}^*$-image.

Note. If Γ is the pseudogroup of example 2, then we give the C^1-topology to Γ_0 and denote this topological pseudogroup by $E_0^d(n)$. The sheaf constructed from $E_0^d(n)$ similarly as $H_*(n)_c$ is denoted by $H_*^d(n)_c$. Then if we set for $(f_1, \cdots, f_n) \in E_0^d(n)$ and $(a_{ij}) \in GL(n, R)$

$$J_0((f_1, \cdots, f_n)) = \left(\frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}\right)(0),$$

$$\iota((a_{ij})) = \iota(\sum_i a_{i1}x_i, \cdots, \sum_i a_{in}x_i),$$

we get the following commutative diagram, where $\bar{J}_0$ is the map induced from J_0. (Cf. [1]).

$$\begin{array}{ccc} & H_*^d(n)_c & \\ {}^{\iota}\nearrow & & \searrow{}^{\bar{J}_0} \\ GL(n, R)_c & \xrightarrow{\text{identity}} & GL(n, R)_c. \end{array}$$

We denote the homeomorphism from $H_*^d(n)_c$ into $H_*(n)_c$ induced from the inclusion by $\iota(=\iota_d)$. If we regard Γ_0 to be a subspace of $E_0^d(n)$, then we can construct a sheaf Γ_{*c}^d which is a subsheaf of $H_*^d(n)_c$. The inclusion from Γ_{*c}^d into $H_*^d(n)_c$ is denoted by $j(=j_\Gamma)$ and set $\iota = \iota_\Gamma = \iota_d \circ j_\Gamma$.

5. *The cohomology set* $H^1(X, \Gamma_{*c})$. Since Γ_{*c} is a subsheaf of $H_*(n)_c$, we can construct a representation $\mathfrak{X}$ of a cocycle of $H^1(X, \Gamma_{*c})$. This $\mathfrak{X}$ is called a Γ-bundle. If X is a normal paracompact topological space with countable open basis, then we obtain the covering homotopy theorem for these bundles. ([8], § 11, [5], § 6). Hence we have

$$(7) \qquad p^*: H^1(X, \Gamma_{*c}) \simeq H^1(X \times I, \Gamma_{*c}),$$
$$i_t^*: H^1(X \times I, \Gamma_{*c}) \simeq H^1(X, \Gamma_{*c}),$$

where I is the $[0, 1]$-interval, p is the projection and i_t is the map given by $i_t(x) = x \times t$, $x \in X$, $x \times t \in X \times I$. We note that (7) is also true for Γ_{*c}^d. By (7) and [1], we get

$$(8) \qquad \bar{J}_0^*: H^1(X, H_*^d(n)_c) \simeq H^1(X, GL(n, R)_c),$$

$$(8)' \qquad \bar{J}_0^*: H^1(X, \Gamma_{*c}^d) \simeq H^1(X, GL(m, C)_c),$$

where Γ is the pseudogroup of example 3. By (8), (8)$'$, we obtain

Lemma 4. *If Γ is the pseudogroup of example 2, then X is an almost Γ-manifold if and only if the tangent microbundle of X is induced from a vector bundle.*

Lemma 4$'$. *If Γ is the pseudogroup of example 3, then a smooth manifold X is an almost Γ-manifold if and only if X is an almost complex manifold.*

We also note that by (4), if Γ is in i_d^*-image, then we get

$$J_0(g_{UV}(x)) = \frac{\partial(g_{UV}(x))}{\partial(a_1, \cdots, a_n)}(0),$$

therefore $\bar{J}_0^*(\tau)$ is the tangent bundle of X if X is a differentiable manifold.

By (7), we also have

Lemma 5. *If X is a paracompact topological manifold, then for any open covering system $\{U\}$ of X, there exists a refinement $\{V\}$ of X such that*

$$H^1(V, \Gamma_{*c})=0. \tag{9}$$

6. *Integrable almost Γ-structures.* By definition, a topological manifold X is an almost Γ-manifold if and only if there exists an element τ_0 of $H^1(X, \Gamma_{*c})$ such that $i^*(\tau_0)=\tau$, the tangent microbundle of X.

Definition. We call τ_0 comes from a Γ-structure of X if X is a Γ-manifold and the class of $\{g_{UV}(x)\}$ in $H^1(X, \Gamma_{*c})$ is τ_0. Here $g_{UV}(x)$ is that of given by (3).

In general, we set

$$\tau_0=\{f_U(x)g_{UV}(x)f_V(x)^{-1}\},\ f_U(x)\in H^0(U, H_*(n)_c), \tag{10}$$
$$f_U(x)g_{UV}(x)f_V(x)^{-1}\in \Gamma_{*x},\ \text{the stalk of } \Gamma_{*c} \text{ at } x.$$

We denote the representation of $f_U(x)$ by $\bar{f}_U(x)$. Here $\bar{f}_U(x)$ need not be defined on $\boldsymbol{R}^n$.

Definition. Let X be an almost Γ-manifold whose almost Γ-structure is determined by τ_0 given by (10). Then we call the almost Γ-structure of X is integrable if and only if

$$\bar{f}_U(x)t_{U,x,y}\bar{f}_U(y)^{-1}\in \Gamma, \tag{11}$$

for all U. Here $t_{U,x,y}(=t_{U,x}t_{U,y}{}^{-1})$ is the pseudoconnection of X.

By definition, the integrability of an almost Γ-structure is determined by the equivalence class of τ_0 as a Γ-bundle.

Using lemma 5, we can prove (cf. [4], [6]),

Theorem 2. *τ_0 comes from a Γ-structure of X if and only if the almost Γ-structure of X defined by τ_0 is integrable.*

Corollary 1. *If $H_*(n)_c/i^*(\Gamma_{*c})$ becomes a constant sheaf of discrete groups, then an almost Γ-manifold admits a Γ-structure.*

We denote by $SE_0(n)$ the connected component of the identity of $E_0(n)$, and the sheaf constructed from $SE_0(n)$ similarly as $H_0(n)_*$ is denoted by $SH_*(n)_c$. Then we get by corollary 1,

Corollary 2. *X is a stable manifold (cf. [3]) if and only if the tangent microbundle of X can be regarded to be an element of $H^1(X, SH_*(n)_c)$.*

Corollary 3. *X is an orientable manifold if and only if the tangent microbundle of X can be regarded to be a Γ-bundle, where Γ is the pseudogroup of example* 1.

7. *Connection of Γ-bundles.* We can define $C^r(X, \Gamma_{*c})$ similarly as $C^r(X, G)$. Then we define a connection form of a Γ-bundle

$\{\varphi_{UV}(x)\}$ to be a collection $\{s_U\}$, $s_U \in C^1(U, \Gamma_{*c})$ such that

(5)′ $$\varphi_{UV}(x_0)^{-1}s_U(x_0, x_1)\varphi_{UV}(x_1)=s_V(x_0, x_1).$$

By (8), if $\{\varphi_{UV}(x)\}$ is an $E_0^d(n)$-bundle, then $\{\varphi_{UV}(x)\}$ has a connection form. Therefore, by theorem 3 of [2]-I, we obtain

Theorem 3. *If X is a paracompact, simply connected topological manifold, then X has an almost Γ-structure if and only if the tangent microbundle of X has a connection form with matrix-valued curvature form. Here Γ is the pseudogroup of example 2.*

Theorem 3′. *If X is a paracompact, simply connected smooth manifold, then X has an almost complex structure if and only if the tangent microbundle of X has a connection form with $GL(m, C)$-valued curvature form.*

On the other hand, since the sequence

$$H^0(X, \Gamma_{*c})\longrightarrow H^0(X, H_*(n)_c/i*(\Gamma_{*c}))\xrightarrow{\delta}$$
$$\longrightarrow H^1(X, \Gamma_{*c})\xrightarrow{i*}H^1(X, H_*(n)_c),$$

is exact, we may compute the set of equivalence classes of almost Γ-structures on X.

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