103. *Axiom Systems of B-Algebra.* VI

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In this note, by an algebraic formulation of the classical propositional calculus axiom systems given by Frege (see, [6]) and Russell (see, [4]), we shall give new axiom systems which are equivalent to the B-algebra defined by K. Iséki (see, [2]).

Let $M=\langle X, 0, *, \sim\rangle$ be an abstract algebra satisfying the following axioms:

F 1 $x*y\leqslant x$.

F 2 $(x*y)*(y*z)\leqslant(x*y)*z$.

F 3 $\sim x*\sim y\leqslant y*x$.

F 4 $x\leqslant\sim(\sim x)$.

F 5 $\sim(\sim x)\leqslant x$.

F 6 $0\leqslant x$.

D 1 If $x\leqslant y$ and $y\leqslant x$, then we define $x=y$.

D 2 $x\leqslant y$ means $x*y=0$.

Then the abstract algebra M is called a B-algebra (for details, see, [2]).

Consider the following axiom systems.

(1) $x*y\leqslant x$.

(2) $(x*z)*(y*z)\leqslant(x*y)*z$.

(3) $x*y\leqslant\sim y*\sim x$.

(4) $0\leqslant x$.

(5) $x\leqslant y$ and $y\leqslant x$ imply $x=y$.

(3′) $\sim x*y\leqslant\sim y*x$.

(3″) $x*\sim y\leqslant y*\sim x$.

(3‴) $\sim x*\sim y\leqslant y*x$.

According to the definition given by K. Iséki, if $\langle X, 0, *, \sim\rangle$ satisfies axioms (1), (2), (3), ((3′), (3″), (3‴)), (4), (5), and (6) it is called a B (NB, BN, NBN)-algebra respectively.

First we shall prove that F1—F6 axioms system is a B-algebra.

By axioms F4, F5, and D1, we have

7 $\sim(\sim x)=x$.

In axiom F3, if we substitute $\sim x$ for x and $\sim y$ for y, then we have $\sim(\sim x)*\sim(\sim y)\leqslant\sim y*\sim x$. Hence by 7, we have

8 $x*y\leqslant\sim y*\sim x$.

F1, F2, 8, F6, D1, D2 hold in F1—F6 axioms system, it is a B-algebra.

Next we shall give a proof that the B-algebra satisfies $F1$—$F6$. For proofs, we freely use some powerful results in K. Iséki's paper (see [1]).

His results are read as:

Lemma 1. *Any NB-algebra (or BN-algebra) is an NBN-algebra.*

Lemma 2. *Any B-algebra is an NB-algebra and a BN-algebra.* $\sim(\sim x)=x$ *holds in B-algebra.*

By Lemma 2, $x\leqslant\sim(\sim x)$ and $\sim(\sim x)\leqslant x$ hold in B-algebra.

By Lemma 1 and 2, any B-algebra is an NBN-algebra, then $\sim x*\sim y\leqslant y*x$. The proof is complete.

Further we shall prove that the following $R1$—$R7$ axioms system is equivalent to the B-algebra defined by K. Iséki (see [3]). He has proved that a B-algebra is equivalent to the following $H1$—$H5$ axioms system (see, [5]).

$H\,1$ $x*y\leqslant x$.

$H\,2$ $(x*y)*(x*z)\leqslant z*y$.

$H\,3$ $(x*y)*(z*y)\leqslant x*z$.

$H\,4$ $x*\sim y\leqslant y$.

$H\,5$ $x*(x*\sim y)\leqslant x*y$.

$R1$—$R7$ axioms system is given as follows:

$R\,1$ $x*y\leqslant x$.

$R\,2$ $(x*y)*(x*z)\leqslant z*y$.

$R\,3$ $(x*y)*z\leqslant(x*z)*y$.

$R\,4$ $x\leqslant\sim(\sim x)$.

$R\,5$ $\sim x\leqslant\sim x*x$.

$R\,6$ $\sim x*y\leqslant\sim y*x$.

$R\,7$ $0\leqslant x$.

$D\,1$ $x\leqslant y$ means $x*y=0$.

$D\,2$ If $x\leqslant y$ and $y\leqslant x$, then we define $x=y$.

In $R3$, we substitute $x*y$, $z*y$, and $x*z$ for x, y, and z respectively, then the right side is identical with $R2$. Hence by 7, we have

8 $(x*y)*(z*y)\leqslant x*z$.

In $R3$, putting $y=x$ and $z=(x*y)*x$, we have $(x*x)*((x*y)*x)\leqslant(x*((x*y)*x))*x$. By $R1$ the right side is equal to 0. Hence we have $x*x=(x*y)*x$. Therefore by $R1$ we have

9 $x*x=0$.

Next we shall prove the converce. It is proved by Prof. K. Iséki that in any B-algebra hold the syllogistic law, the commutative law, $x=\sim(\sim x)$ and $\sim x*y\leqslant\sim y*x$. These are $R2$, $R3$, $R4$, and $R6$.

Further in any B-algebra hold the followings (see, [3]):

a) $x*x=0$.

b) $x*(x*(\sim y))\leqslant x*y$.

In b), if we put $x=\sim x$, $y=\sim x$, then we have $\sim x*(\sim x*(\sim(\sim x))\leqslant \sim x*\sim x$. Hence we have $\sim x*(\sim x*\sim(\sim x))=0$. This means

c) $\sim x\leqslant \sim x*x$. The proof is complete.

If we put $x=\sim x$ and $y=x$ in $R\,6$, then we have $\sim(\sim x)*x\leqslant \sim x*\sim x$. The right side is equal to 0 by 9, Hence by 7, we have $\sim(\sim x)*x=0$. Therefore we have

10 $\sim(\sim x)\leqslant x$. By $R\,4$, 10, and $D\,2$, we have the following

11 $\sim(\sim x)=x$.

In $R\,6$, if we put $x=\sim x$, $y=\sim y$, then by 11, we have

12 $x*\sim y\leqslant y*\sim x$.

In 8, put $z=z*(\sim y)$, $x=y*\sim z$, $y=z$, and we have $((y*\sim z)*z)*((z*\sim y)*z)\leqslant (y*\sim z)*(z*\sim y)$. The right side is equal to 0 by 12, and further the second term of the left side is equal to 0 by $R\,1$. Hence we have $(y*\sim z)*z=0$. Therefore

13 $x*\sim y\leqslant y$.

From axiom 2 i.e., the logical syllogistic law, we have the following. If $z\leqslant y$, then $x*y\leqslant x*z$, i.e., $z*y=0$ implies $(x*y)*(z*x)=0$. In the above, we put $z=z*\sim y$, $y=y*\sim z$, by 12 we have

14 $x*(y*\sim z)\leqslant x*(z*\sim y)$.

In $R\,5$ we substitute $\sim x$ for x, then we have $\sim(\sim x)\leqslant \sim(\sim x)*\sim x$, and further by 11 we have

15 $x\leqslant x*\sim x$.

In the syllogistic law, if we put $y*z$ into z, and $(w*z)*(w*y)$ into y, then by $R\,2$ we have

16 $x*(y*z)\leqslant x*((w*z)*(w*y))$.

In 6, if we put $x=(x*(y*z))*(x*(y*w))$, $y=z$, $z=w$, $w=y$, then we have $((x*(y*z))*(x*(y*w)))*(z*w)\leqslant((x*(y*z))*(x*(y*w)))*((y*w)*(y*z))$. The right side is equal to 0, because it is identical with $R\,2$ substituted $y*z$ for y and $y*w$ for z. Hence by $R\,7$ we have

17 $(x*(y*z))*(x*(y*w))\leqslant z*w$.

Let us put $x=x*(y*z)$, $y=z*w$, $z=x*(y*w)$ in $R\,3$, then by 17 we have $((x*(y*z))*(z*w))*(x*(y*w))\leqslant 0$. Hence by $R\,7$ and $D\,1$, we have

18 $(x*(y*z))*(z*w)\leqslant x*(y*w)$.

In the above, if we put $y=x$, $z=y$, $w=\sim x$, then we have $x*(x*\sim x)$ as the right side, and $(x*(x*y))*(y*\sim x)$ as the left side. On the other hand by 15 the former is equal to 0, then by $R\,7$ we have $(x*(x*y))*(y*\sim x)=0$. By $D\,1$ we have

19 $x*(x*y)\leqslant y*\sim x$.

Putting $x=x*(x*y)$, $y=x$, $z=y$, in 14, then we have $(x*(x*y))*$

$(x*\sim y)\leqslant(x*(x*y))*(y*(\sim x))$. The right side is equal to 0 by 19. Therefore by $R7$ and $D1$ we have

20 $x*(x*y)\leqslant x*(\sim y)$.

Putting $\sim y$ into y in the above, we have $x*\sim(\sim y)$ as the right side, and further by 11 $\sim(\sim y)=y$. At the same time the left side is $x*(x*\sim y)$. Hence we have

21 $x*(x*\sim y)\leqslant x*y$.

Axioms $R1$, $R2$, theses 8, 13, and 21 are $H1$—$H5$.

References

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