99. *Obstructions to Locally Flat Embeddings of Bounded Combinatorial Manifolds*

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In the paper [1], H. Noguchi showed that for any proper $(p+1)$-flat embedding $f: M\to W$, where M is an oriented closed n-manifold and W is an oriented $(n+2)$-manifold without boundary, the p-homology class Ω_f of M, called the Whitehead class of f, is defined, and if $\Omega_f=0$, the embedding f can be arbitrarily approximated by a p-flat embedding $g: M\to W$, $0\leq p\leq n-2$.

We will extend this for bounded manifolds M and W as follows.

Let M be a compact oriented n-manifold with non-vacuous boundary ∂M, and W be an oriented $(n+2)$-manifold with non-vacuous boundary ∂W. Let $f: M\to W$ be a proper embedding; that is to say, $f(\text{Int}\, M)\subset \text{Int}\, W$ and $f(\partial M)\subset\partial W$. Then, by § 4 of [1], f is $(n-1)$-flat. Hence it is assumed that f is a $(p+1)$-flat embedding, $0\leq p\leq n-2$.

Next we define the p-homology class $\Omega_f\in H_p(M, \partial M; G^{n-p-1})$ of M mod ∂M, called the Whitehead class of the embedding f, where G^{n-p-1} is the knot cobordism group of dimension $n-p-1$. In fact by Theorem 3 of [2] (see § 1 of [1]), the class Ω_f is invariant under the iso-neighboring relation of proper embeddings of M in W.

The main result of the paper is as follows.

Theorem. *If the Whitehead class Ω_f of f is the identity, f can be arbitrarily approximated by a p-flat embedding.*

If C is an n-cell, then $H_p(C, \partial C; G^{n-p-1})=0$ for $0\leq p\leq n-2$, and we have the following.

Corollary 1. *Let C, D be n-, $(n+2)$-cells and $f: C\to D$ be a proper embedding. Then f is arbitrarily approximated by a locally flat embedding.*

Since $H_0(M, \partial M; G^{n-1})=0$ for each manifold M with non-vacuous boundary ∂M, we have the following.

Corollary 2. *Any 1-flat proper embedding $f: M\to W$ can be arbitrarily approximated by a locally flat embedding.*

From now on it will be assumed that the embedding $f: M\to W$ is $(p+1)$-flat.

Notation. Let $\varphi: K\to L$ be a triangulation of f. Then ∂K means a subcomplex of K covering ∂M, and Int K means the set of simplexes $K-\partial K$. Let $\triangle$ be an oriented r-simplex of ∂K. Then $\nabla_\partial(\square_\partial)$ is an

$(n-r-1)$-cell ($(n-r+1)$-cell) dual to $\triangle(\varphi\triangle)$ in $\partial K(\partial L)$ whose orientation is induced from $\partial\nabla(\partial\square)$ (for $\nabla, \square$ see [1]). The barycenter of $\triangle$ will be denoted by c.

Lemma 1. *For each oriented r-simplex $\triangle$ of ∂K, the pair $\partial(f\nabla, \square)-\mathrm{Int}\,(f\nabla_\partial, \square_\partial)$ is homeomorphic to $(fLk(\triangle, K), Lk(f\triangle, L))$, and is flat if $r>p$ and $(p-r)$-flat if $r\leq p$.*

Proof. Since $(f\nabla, \square)$ is homeomorphic to the join $(fLk(\triangle, K), Lk(f\triangle, L))*fc$, $\partial(f\nabla, \square)$ is homeomorphic to $\partial(fLk(\triangle, K), Lk(f\triangle, L))*fc \cup (fLk(\triangle, K), Lk(f\triangle, L))$. Since $\partial(fLk(\triangle, K), Lk(f\triangle, L))$ is homeomorphic to $(fLk(\triangle, \partial K), Lk(f\triangle, \partial L))$, $\partial(fLk(\triangle, K), Lk(f\triangle, L))*fc$ is homeomorphic to $(f\nabla_\partial, \square_\partial)$. Hence $\partial(f\nabla, \square)-\mathrm{Int}\,(f\nabla_\partial, \square_\partial)$ is homeomorphic to $(fLk(\triangle, K), Lk(f\triangle, L))$. Let x be an interior point of $\triangle$, then $Lk(x, \varphi)$ is homeomorphic to $\partial(\varphi\triangle)*(fLk(\triangle, K), Lk(f\triangle, L))$. The nthe last half of the lemma follows from the argument of Lemma 11 of 1.

Definition. In the paper a knot is a locally flat sphere pair and a node is a locally flat cell pair, see [1]. Let $\triangle_i$ be an oriented p-simplex of Int K, then $\partial(f\nabla_i, \square_i)$ is an $(n-p-1)$-knot by Lemma 11 of [1]. By κ_i we denote the knot cobordism class of $\partial(f\nabla_i, \square_i)$. Then we have a p-chain

$$\omega=\sum_i \kappa_i\triangle_i$$

of K mod ∂K with the $(n-p-1)$-knot cobordism group G^{n-p-1} as the coefficient group, where $\triangle_i$ ranges over the p-simplexes of Int K.

It is shown by Lemmas 12 and 14 of [1] that this p-chain is a p-cycle of K mod ∂K and that the homology class $\Omega_f \in H_p(M, \partial M; G^{n-p-1})$ of ω is invariant under the subdivision $\varphi: K\to L$, and is an invariant of the iso-neighboring relation. We call Ω_f *the Whitehead class* of the embedding f of M in W.

Lemma 2. *Let (S, T) be a sphere pair such that $(S, T)=(C_1, D_1)\cup(C_2, D_2)$, where C_i, D_i are $m, (m+2)$-cells, $i=1, 2$, and $(C_1, D_1)\cap(C_2, D_2)=\partial(C_1, D_1)=-\partial(C_2, D_2)$. If (C_1, D_1) is a node, there is a knot $(\tilde{S}, T)$ such that $(\tilde{S}, T)\cap D_1=(\tilde{S}\cap D_1, T\cap D_1)=(C_1, D_1)$ and $(\tilde{S}, T)$ is knot cobordant to zero.*

Proof. Let (C_1', D_1') be a copy of (C_1, D_1), and identify D_1' with $-D_2$ in such a way that $-\partial(C_1', D_1')$ and $\partial(C_2, D_2)$ are identified. Then $(C_1, D_1)\cup(-(C_1', -D_2))=(C_1\cup(-C_1'), T)=(S', T)$ is a knot by the proof of Lemmas 6 and 7 of [1]. Let κ be a knot cobordism class of (S', T) and x a point of Int $(-C_1')$. Then $St(x, (S', T))=(C_3, D_3)$ is flat. By an argument similar to the one above, we may cut (C_3, D_3) from (S', T) and glue to (S', T) a node (C_3', D_3) with boundary $\partial(C_3, D_3)$, where $(C_3', D_3)\cup\partial(C_3', D_3)*y$ is a knot representing $-\kappa$. Then $((S', T)-\mathrm{Int}\,(C_3, D_3))\cup(C_3', D_3)=(\tilde{S}, T)$ is knot cobordant to zero by Lemma 10 of [1], and $(\tilde{S}, T)$ is the required knot.

Lemma 3. *Let $g: M \to W$ be a $(p+1)$-flat embedding. Let $\triangle$ be an oriented p-simplex of $\partial\tilde{K}$, where $\tilde{\varphi}: \tilde{K} \to \tilde{L}$ is a triangulation of g. Then there is an $(n-p)$-node $(\tilde{\nabla}, \square)$ such that $\partial(\tilde{\nabla}, \square) \cap (\partial\square - \mathrm{Int}\,\square_\partial) = \partial(g\nabla, \square) - \mathrm{Int}\,(g\nabla_\partial, \square_\partial)$, and we have an embedding $h_\triangle: M \to W$ such that*

(1) $h_\triangle(M) = (g(M) - \mathrm{Int}\,g(\partial\triangle * \nabla)) \cup \mathrm{Int}\,(g\partial\triangle * \tilde{\nabla})$

(2) $h_\triangle \mid M - \partial\triangle * (\mathrm{Int}\,\nabla \cup \mathrm{Int}\,\nabla_\partial) = g \mid M - \partial\triangle * (\mathrm{Int}\,\nabla \cup \mathrm{Int}\,\nabla_\partial)$

(3) *$h_\triangle$ is flat at each point of $\partial\triangle * \nabla - \partial\triangle$.*

Consequently, $h_\triangle$ is flat at a point x of $\mathrm{Int}\,M$ *if g is.*

Proof. Let $\partial(g\nabla, \square) - \mathrm{Int}\,(g\nabla_\partial, \square_\partial) = (C, D)$; then $\partial(g\nabla, \square) = (C, D) \cup (g\nabla_\partial, \square_\partial)$, and $(C, D) \cap (g\nabla_\partial, \square_\partial) = \partial(C, D) = -\partial(g\nabla_\partial, \square_\partial)$. By Lemma 1, (C, D) is an $(n-p-1)$-node, and by Lemma 2, we have a knot $(\partial\tilde{\nabla}, \partial\square)$ which is knot cobordant to zero, and such that $(\partial\tilde{\nabla}, \partial\square) \cap (\partial\square - \mathrm{Int}\,\square_\partial) = \partial(g\nabla, \square) - \mathrm{Int}\,(\nabla_\partial, \square_\partial)$. Then we have a node $(\tilde{\nabla}, \square)$ with boundary $(\partial\tilde{\nabla}, \partial\square)$ that has the required property.

The construction of the required embedding $h_\triangle$ using $\tilde{\nabla}$ is the same as Lemma 15 of [1], and so we will omit the proof.

Proof of theorem. For a given ε-neighborhood of fM in W, we subdivide K, L so fine that the diameter of the star of every simplex of φK in L is smaller than ε, where $\varphi: K \to L$ is a triangulation of f. Let $\omega = \sum_i \kappa_i \triangle_i$ be the p-cycle mod ∂M obtained from φ. By the assumption $\Omega_f = 0$, there is a $(p+1)$-chain γ of K such that $\partial\gamma = \omega + \beta$, where β is a p-chain of ∂K. Then by the argument of the proof of the main Theorem of [1], we have an embedding $g: M \to W$ such that g is flat at $x \in (M - |K^p|) \cup \bigcup_\triangle \mathrm{Int}\,\triangle$, where $\triangle$ is a p-simplex of $\mathrm{Int}\,K$. Then by Lemma 15 of [1], g is $(p+1)$-fiat, and is fiat at $x \in (M - |\tilde{K}^p|) \cup \bigcup_{\tilde{\triangle}} \mathrm{Int}\,\tilde{\triangle}$, where $\tilde{\triangle}$ is a p-simplex of $\mathrm{Int}\,K$ and $\tilde{\varphi}: \tilde{K} \to \tilde{L}$ is a triangulation of g and $\tilde{K}$ is a subdivision of K. Define h by taking $h \mid \partial\triangle * \nabla = h_\triangle \mid \partial\triangle * \nabla$, and $h \mid (M - \bigcup_\triangle \mathrm{Int}\,(\partial\triangle * \nabla)) = g \mid (M - \bigcup_\triangle \mathrm{Int}\,(\partial\triangle * \nabla))$, where $\triangle$ ranges over the p-simplexes of $\partial\tilde{K}$ and $h_\triangle$ is the embedding obtained in Lemma 3. Then h is a p-flat embedding, proving Theorem.

References

[1] H. Noguchi: Obstructions to locally flat embeddings of combinatorial manifolds (to appear).

[2] J. F. P. Hudson and E. C. Zeeman: On regular neighbourhoods, Proc.London Math. Soc., **14**, 719-745 (1964).