98. *On Kernels of Invariant Functional Spaces*

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Introduction. Deny introduced in [6] the notion of invariant functional spaces and he proved that to an invariant functional space $\mathfrak{X}$ corresponds a convolution kernel κ in the following sense: each potential u_f in $\mathfrak{X}$ generated by a bounded measurable function f with compact support is equal to the convolution $\kappa * f$. In this paper, we shall prove that the converse is valid. That is, for a positive measure κ of positive type, there exists an invariant functional space with kernel κ. Furthermore we shall give a necessary and sufficient condition for a positive measure κ of positive type to be the kernel of a special Dirichlet space.

1. Invariant functional spaces. Let X be a locally compact abelian group. We denote by dx the Haar measure of X. We define two kinds of functional spaces on X.

Definition 1. *A weak invariant functional space $\mathfrak{X}=\mathfrak{X}(X)$ with respect to X and dx is a Hilbert space of real valued locally summable functions satisfying the following two conditions.*

(1.1) *For any compact subset K in X, there exists a positive constant $A(K)$ such that*

$$\left|\int_K u(x)dx\right| \leq A(K)||u||$$

for any u in $\mathfrak{X}$.

(1.2) *Let $U_x u$ be a function obtained from u in $\mathfrak{X}$ by the translation $x \in X$. For any u in $\mathfrak{X}$ and any x in X, $U_x u$ is in $\mathfrak{X}$ and $||U_x u|| = ||u||$.*

Two functions which are equal $p.p.$[1] in X represent the same element in $\mathfrak{X}$. By the condition (1.1), for any compact subset K in X, there exists an element u_K in $\mathfrak{X}$ such that

$$(u, u_K) = \int_K u(x)dx$$

for any u in $\mathfrak{X}$. Especially when $u_K(x) \geq 0$ $p.p.$ in X for any compact subset K, $\mathfrak{X}$ is called a positive weak invariant functional space on X.

Definition 2.[2] *A weak invariant functional space $\mathfrak{X}$ is called*

1) A property is said to hold $p.p.$ in a subset E in X if the property holds in E except a set which is locally of measure zero.

2) Cf. [6], p. 12.

an invariant functional space on X if the following additional condition is satisfied.

(2.1) *For any compact subset K in X, there exists a positive constant $A(K)$ such that*

$$\int_K |u(x)|dx \leq A(K)||u||$$

for any u in $\mathfrak{X}$.

Let $\mathfrak{X}$ be an invariant functional space on X. By the condition (2.1) in the above definition, for any bounded measurable function f with compact support, there exists an element u_f in $\mathfrak{X}$ such that

$$(u, u_f) = \int u(x)f(x)dx$$

for any u in $\mathfrak{X}$. This element u_f is called the potential generated by f.[3] Especially when $u_f(x) \geq 0$ *p.p.* in X for any positive bounded measurable function f with compact support, $\mathfrak{X}$ is said to be positive.

Similarly as Aronszajn and Smith [1], we obtain the following lemma.

Lemma 1. *Let $\mathfrak{X}$ be a positive weak invariant functional space on X. For each u in $\mathfrak{X}$, there exists an element $\tilde{u}$ in $\mathfrak{X}$ such that* $|u(x)| \leq \tilde{u}(x)$ *p.p. in X and* $||u|| \geq ||\tilde{u}||$.

Proof. Let P be a closed convex cone in $\mathfrak{X}$ with vertex 0 generated by the set $\{u_K \in \mathfrak{X}; K \text{ is compact in } X\}$. Let u' and u'' be the projections of u and $-u$ to P, respectively. Put

$$\tilde{u} = u' + u''.$$

Then similarly as Aronszajn and Smith did, we see that $\tilde{u}$ satisfies all the required conditions.

By the above lemma, we obtain the following

Lemma 2. *Let $\mathfrak{X}$ be a positive weak invariant functional space on X. Then $\mathfrak{X}$ is a positive invariant functional space on X.*

Proof. It is sufficient to prove that the condition (2.1) is satisfied. By Lemma 1, for any u in $\mathfrak{X}$,

$$\int_K |u(x)|dx \leq \int_K \tilde{u}(x)dx \leq A(K)||\tilde{u}|| \leq A(K)||u||$$

for any compact subset K in X. Hence the condition (2.1) is satisfied and the proof is completed.

Our first theorem concerns with the converse of Deny's theorem mentioned in the introduction.

Theorem 1. *Let X be a locally compact abelian group. For any positive measure κ of positive type in X, there exists a positive invariant functional space with kernel κ.*

Proof. By Lemma 2, it is sufficient to prove that for a positive measure κ of positive type in X, there exists a positive weak invariant

3) Cf. [3], p. 209.

functional space $\mathfrak{X}$ with kernel κ. Put

$\mathfrak{X}'=\{\kappa*f;$ *f is a bounded measurable function with compact support*$\}$.

Then $\mathfrak{X}'$ is a pre-Hilbert space with norm $||u_f||^2=\kappa*f*\check{f}(0)$, where $u_f=\kappa*f$ and $\check{f}(x)=f(-x)$. And we have

$$\left|\int_K u_f(x)dx\right|=|(u_f,\, u_{c_K})|\leq||u_{c_K}||\cdot||u_f||$$

for any compact subset K in X, where $c_K(x)$ is the characteristic function of K. By the above inequality, each fundamental sequence (u_{f_n}) in $\mathfrak{X}'$ is fundamental in the weak topology in $L^1(K)$ for any compact subset K in X. Since $L^1(K)$ is weakly complete,[4] there exists a function u defined $p.p.$ in X such that (u_{f_n}) converges weakly to u in $L^1(K)$ for any compact subset K in X. Furthermore we have

$$\left|\int_K u(x)dx\right|\leq||u_{c_K}||\lim_{n\to\infty}||u_{f_n}||.$$

Let us define the norm of u by

$$||u||=\lim_{n\to\infty}||u_{f_n}||.$$

Then the completion $\mathfrak{X}$ of $\mathfrak{X}'$ is a Hilbert space of locally summable functions and satisfies the condition (1.1) in Definition 1. We shall prove that $\mathfrak{X}$ satisfies the condition (1.2). For any x in X,

$$U_x u_f(y)=u_f(y-x)=\int f(y-x-z)d\kappa(z)=u_{U_x f}$$

for any finite continuous function f with compact support. Hence for any u in $\mathfrak{X}$ and any x in X,

$$U_x u\in\mathfrak{X} \quad \text{and} \quad ||U_x u||=||u||.$$

Thus the condition (1.2) is satisfied and the proof is completed.

2. Special Dirichlet spaces. In this section, we shall consider the kernel of a special Dirichlet space.[5] Choquet and Deny [4] showed that a positive measure κ of positive type is the kernel of a special Dirichlet space D on a locally compact abelian group X if and only if κ is "*le noyaux associé*".[6] We give the other characterization for κ to be the kernel of a special Dirichlet space on X.

Theorem 2. *Let X be a locally compact abelian group. A positive measure κ of positive type in X is the kernel of a special Dirichlet space D on X if and only if κ satisfies the following condition* (*).

(*). *There exists a base of compact neighborhoods $\mathfrak{U}$ of 0 such that for any v in $\mathfrak{U}$, there exists a positive measure σ_v satisfying that*

4) Cf. [8], p. 121.
5) Cf. [3], p. 215.
6) Cf. [4], p. 4261.

(1) $$\kappa \gneqq \kappa * \sigma_v \quad in\ X,$$

(2) $$\kappa = \kappa * \sigma_v \quad in\ Cv,$$

(3) $$\int d\sigma_v \leq 1.$$

Proof. The "only if" part follows from the existence of balayaged measures of the unit measure ε at O.[7] We shall prove the converse. For any v in $\mathfrak{U}$, put

$$\eta_v = \kappa * (\varepsilon - \sigma_v).$$

Then κ being symmetric,

$$\eta_{v_1} * (\varepsilon - \check{\sigma}_{v_2}) = \check{\eta}_{v_2} * (\varepsilon - \sigma_{v_1})$$

for any couple of v_1 and v_2 in $\mathfrak{U}$, where the symbol $\vee$ is the same as in the proof of Theorem 1. Hence

$$\hat{\eta}_{v_1}(\hat{x})(1 - \overline{\hat{\sigma}_{v_2}(\hat{x})}) = \overline{\hat{\eta}_{v_2}(\hat{x})}(1 - \hat{\sigma}_{v_1}(\hat{x}))$$

in $\hat{X}$, where the symbol $\wedge$ over a measure represents the Fourier transform and $\hat{X}$ is the dual group of X. Put

$$\lambda(\hat{x}) = \frac{1 - \hat{\sigma}_{v_1}(\hat{x})}{\hat{\eta}_{v_1}(\hat{x})} = \frac{1 - \overline{\hat{\sigma}_{v_2}(\hat{x})}}{\overline{\hat{\eta}_{v_2}(\hat{x})}}, \tag{i}$$

when $\hat{\eta}_{v_1}(\hat{x})$ and $\hat{\eta}_{v_2}(\hat{x})$ don't vanish. Then $\lambda(\hat{x})$ is real valued. For any v in $\mathfrak{U}$, $\eta_v(\hat{O}) \neq 0$, because $\eta_v \neq 0$. Put

$$\eta'_v = \eta_v / \int d\eta_v.$$

Then η'_v converges vaguely to ε and the support of η'_v tends to $\{O\}$ as v tends to $\{O\}$. That is, $\hat{\eta}_v(\hat{x})/\hat{\eta}_v(\hat{O})$ converges uniformly to 1 in the wide sense. Therefore $\lambda(\hat{x})$ is defined everywhere in $\hat{X}$ and

$$\lambda(\hat{x}) = \lim_{v \to \{O\}} \frac{1 - \hat{\sigma}_v(\hat{x})}{\hat{\eta}_v(\hat{O})}.$$

Hence $\lambda(\hat{x})$ is negative definite function in $\hat{X}$,[8] because the total mass of σ_v is less than or equal to 1 for any v in $\mathfrak{U}$. By (i), $\hat{\kappa}$ is a function defined $p.p.$ in $\hat{X}$ and

$$\lambda(\hat{x})\hat{\kappa}(\hat{x}) = 1$$

$p.p.$ in $\hat{X}$, because $\hat{\sigma}_v(\hat{x}) \neq 1$ $p.p.$ in $\hat{X}$. That is, $\lambda(\hat{x})^{-1}$ is locally summable. Consequently by Beurling and Deny's theorem,[9] there exists a special Dirichlet space with kernel κ. This completes the proof.

Remark. If κ is "*le noyaux associé*", it is obvious that κ satisfies the condition $(*)$ in Theorem 2.

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7) Cf. [7], Lemma 10.

8) Cf. [5], pp. 9-11.

9) Cf. [3], p. 215 and [5], pp. 12-13.

References

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