97. *Ideals and Homomorphisms in Some Near-Algebras*

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§1. A real vector space $\mathcal{A}$ is called a *near-algebra* if, for any pair of elements f and g in $\mathcal{A}$, the product fg is defined and satisfies the following two conditions:

(1) $(fg)h=f(gh)$; (2) $(f+g)h=fh+gh$ for f, g, and h in $\mathcal{A}$.

The left distributive law: $h(f+g)=hf+hg$ is not assumed. Therefore, a near-algebra is a *near-ring* which has been defined in [6, pp. 71–74].

Let E be a real Banach space. Let f and g be mappings of E into E. We define the linear combination $\alpha f+\beta g$ (α and β are real numbers) by

$$(\alpha f+\beta g)(x)=\alpha f(x)+\beta g(x) \quad \text{for every } x\in E,$$

and the product fg by

$$(fg)(x)=f(g(x)) \quad \text{for every } x\in E.$$

Let $\mathcal{A}$ be a near-algebra of mappings of E into E. A subset I of $\mathcal{A}$ is said to be *an ideal* if it satisfies the following two conditions:

(1) *I is a linear subset of $\mathcal{A}$;*

(2) *$f\in I$, $g\in\mathcal{A}$ imply fg, $gf\in I$.*

The ideals of *distributively generated* near-rings have been studied by [2] and [3]. Obviously, near-algebras of mappings on Banach spaces are, in general, not distributively generated.

Examples (cf. [4] and [5]). 1. A mapping f of E into E is said to be *constant if*

$$f(x)=a \quad \text{for every } x\in E$$

for a fixed element $a\in E$. We denote this mapping f by c_a. Since

$$\alpha c_a+\beta c_b=c_{\alpha a+\beta b} \quad \text{and} \quad c_a c_b=c_a,$$

the set $I(E)$ of all constant mappings on E is a near-algebra. It is obvious that, if a near-algebra $\mathcal{A}$ contains $I(E)$, $I(E)$ is a minimal ideal of $\mathcal{A}$, and that $\mathcal{A}$ has no proper non-zero ideal if and only if $\mathcal{A}=I(E)$.

2. Let $\mathcal{A}$ be a near-algebra whose elements are bounded (transform every bounded set into a bounded set) and continuous mappings of E into E. Then, the set $\mathcal{A}(C)$ of compact (transform every bounded set into a compact set) and continuous mappings in $\mathcal{A}$ is an ideal of $\mathcal{A}$.

§2. Let I be an ideal of a near-algebra $\mathcal{A}$. Let us write

$f \sim g(I)$ if $f-g \in I$. Then, this relation satisfies

(1) $f \sim g(I)$ implies $f+h \sim g+h(I)$ for any $h \in \mathcal{A}$;

(2) $f \sim g(I)$ implies $fh \sim gh(I)$ for any $h \in \mathcal{A}$.

However, this relation does not satisfy the following condition:

(3) $f \sim g(I)$ implies $hf \sim hg(I)$ for any $h \in \mathcal{A}$.

In [1], it was shown that this relation satisfies the conditions (1), (2), and (3) if and only if the set I satisfies the following three conditions:

(1) *I is a linear subset of $\mathcal{A}$;*

(2) *$f \in I$, $g \in \mathcal{A}$ imply $fg \in I$;*

(3) *$f \in I$, g, $h \in \mathcal{A}$ imply $g(f+h)-gh \in I$.*

A subset I which satisfies these three conditions is called *an NA-ideal.*

Example (cf. [4] and [5]). **1.** Let $\mathcal{A}$ be a near-algebra of all bounded and continuous mappings on E. The set $I(E)$ defined in the previous section is not an NA-ideal of $\mathcal{A}$, although it is a minimal ideal of $\mathcal{A}$.

2. A mapping f of E into E is said to be (Fréchet-) *differentiable* if, for any $a \in E$, there exists a bounded linear mapping l of E into E such that

$$f(a+x)-f(a)=l(x)+r(a, x) \text{ for every } x \in E \text{ where } \lim_{||x|| \to 0} \frac{r(a, x)}{||x||}=0.$$

This linear mapping l may depends on a and is denoted by $f'(a)$. Let $\mathcal{A}$ be the set of all differentiable mappings f such that $f(0)=0$. Then, $\mathcal{A}$ is a near-algebra and the set $\{f \in \mathcal{A} \mid f'(0)=0\}$ is an NA-ideal of $\mathcal{A}$.

The purpose of this paper is to make clear the relation between these two kinds of ideals. We continue to assume that E is a Banach space, although the discussions involved are sometimes purely algebraic.

§ 3. We begin with a lemma which plays an important rôle in the following discussions. In the sequel, we denote by L the Banach algebra of all bounded linear mappings on E.

Lemma 1. *Let $\mathcal{A}$ be a near-algebra of mappings on E. If $\mathcal{A} \supset L$, then we have either $\mathcal{A} \supset I(E)$ or $f(0)=0$ for every $f \in \mathcal{A}$.*

Proof. We prove that, if there exists an element $f \in \mathcal{A}$ such that

$$f(0)=a \neq 0,$$

then $I(E) \subset \mathcal{A}$. At first, if f satisfies this condition, then $c_a \in \mathcal{A}$, because

$$c_a(x)=a=f(0)=f0(x) \quad \text{for every } x \in E,$$

which means that $c_a = f0$.

Next, let b be an arbitrary non-zero element. Then, the linear mapping $b\otimes\bar{a}$ which is defined by

$$(b\otimes\bar{a})(x)=\bar{a}(x)b,$$

where $\bar{a}\in\bar{E}$ (the conjugate space of E) satisfies $\bar{a}(a)=1$, is contained in $\mathcal{A}$ and

$$c_b(x)=b=(b\otimes\bar{a})(a)=(b\otimes\bar{a})c_a(x) \quad \text{for every } x\in E,$$

hence it follows that $c_b=(b\otimes\bar{a})c_a\in\mathcal{A}$.

Now, we can prove the following theorem.

Theorem 1. *Let $\mathcal{A}$ be a near-algebra of mappings on E. If $\mathcal{A}\supset L$, then every NA-ideal is an ideal.*

Proof. Let I be an NA-ideal. We have only to prove that $gf\in I$ if $f\in I$ and $g\in\mathcal{A}$.

Since I is an NA-ideal,

$$g(f+h)-gh\in I \text{ if } f\in I \text{ and } g,\ h\in\mathcal{A}.$$

Putting $h=0$, we have

$$gf-g0\in I.$$

Since I is a linear subset, we have only to prove that $g0\in I$.

(i) If $f(0)=0$ for every $f\in\mathcal{A}$, then $g0=0\in I$.

(ii) If $\mathcal{A}\supset I(E)$ and $g(0)=b\neq 0$, since $c_b=g0$, we have only to prove that $c_b\in I$. Now, for a non-zero element $f\in I$, there exists $y\in E$ such that

$$f(y)=a\neq 0.$$

Let us take $\bar{a}\in\bar{E}$ such that $\bar{a}(a)=1$. Then, for the bounded linear mapping $b\otimes\bar{a}$, we have

$$c_b=(b\otimes\bar{a})c_a=(b\otimes\bar{a})(c_b+c_a)-(b\otimes\bar{a})c_b\in I,$$

because $c_a=fc_y\in I$.

Therefore, by Lemma 1, the proof is completed.

As we have mentioned in the second section, an ideal is not necessarily an NA-ideal. Then, in what cases is every ideal an NA-ideal? It is clear that, in the (near) algebra L, every ideal is an NA-ideal. We have another near-algebra of this kind. Let us consider the set

$$L+I(E)=\{l+c_a \mid l\in L \quad \text{and} \quad a\in E\}.$$

Under the definitions of sum and product given in the first section, this is a near-algebra. Since the left distributive law is still not satisfied, this is not an algebra. Let I be an ideal of this near-algebra $L+I(E)$. Then, for

$$l+c_a\in I \quad \text{and} \quad l_i+c_{b_i}\in L+I(E)\ (i=1,2), \text{ we have}$$

$$\begin{aligned}&(l_1+c_{b_1})((l+c_a)+(l_2+c_{b_2}))-(l_1+c_{b_1})(l_2+c_{b_2})\\&=l_1(l+c_a)+l_1(l_2+c_{b_2})+c_{b_1}-l_1(l_2+c_{b_2})-c_{b_1}\\&=l_1(l+c_a)\in I,\end{aligned}$$

hence it follows that, in $L+I(E)$, every ideal is an NA-ideal.

Conversely, we can prove the following theorem.

Theorem 2. *Let $\mathcal{A}$ be a near-algebra of bounded mappings on E such that $\mathcal{A}\supset L$. If every ideal is an NA-ideal and $I(E)\subset\mathcal{A}$, then $\mathcal{A}=L+I(E)$.*

Proof. Since $I(E)$ is an ideal, it is an NA-ideal. Let f be an arbitrary element of $\mathcal{A}$. Then, by the definition of NA-ideals, we have

$$f(g+c_a)-fg\in I(E) \quad \text{for every } a\in E \text{ and } g\in\mathcal{A}.$$

Since $\mathcal{A}\supset L$, we can replace g by the identity mapping, and we have that

$$f(x+a)-f(x)$$

is constant with respect to x. Putting $x=0$, we have

$$f(x)=f(x+a)-f(a)+f(0) \quad \text{for every } x\in E,$$

which means that

$$f=f_a+c_{f(0)}$$

where $f_a(x)=f(x+a)-f(a)$.

Therefore, we have only to prove that f_a is linear. To prove this, we shall make use of the following equation:

$$f(x+y)=f(x)+f(y)-f(0) \quad \text{for every } x,\ y\in E.$$

Now,

$$\begin{aligned} f_a(x+y)&=f(x+y+a)-f(a)\\ &=f(x+y)-f(0)\\ &=f(x)+f(y)-2f(0)\\ &=(f(x)-f(0))+(f(y)-f(0))\\ &=(f(x+a)-f(a))+(f(y+a)-f(a))\\ &=f_a(x)+f_a(y). \end{aligned}$$

§4. It is natural to conjecture that, *if* (i) $\mathcal{A}\supset L$, (ii) $f(0)=0$ *for every* $f\in\mathcal{A}$ *and* (iii) *every ideal is an NA-ideal, we have* $\mathcal{A}=L$. However, in the case when $f(0)=0$ for every $f\in\mathcal{A}$, we can not make use of the set $I(E)$ which played an essential rôle in the proof of Theorem 2. A standard method to prove this conjecture may be to construct a new near-algebra $\mathcal{A}+I(E)$ (the direct sum); from $\mathcal{A}+I(E)=L+I(E)$ it easily follows that $\mathcal{A}=L$. This method, however, does not serve for our purpose, because, even if the near-algebra $\mathcal{A}$ satisfies the condition that every ideal is an NA-ideal, $\mathcal{A}+I(E)$ does not always satisfy this condition.

Here, we can only give a partial result. We need a lemma.

Lemma 2. *Let $\mathcal{A}$ be a near-algebra of differentiable mappings on a Banach space E. If $\mathcal{A}\supset L$, then $\mathcal{A}=L+D_0$, where $D_0=\{f\in\mathcal{A}\mid f'(0)=0\}$; in other words, for any $f\in\mathcal{A}$ there exists uniquely a pair of elements $l_f\in L$ and $f_0\in D_0$ such that $f=l_f+f_0$.*

Proof. For any $f\in\mathcal{A}$, we have

$$f=f'(0)+(f-f'(0))$$

where $f'(0) \in L$ and $(f-f'(0))'(0)=f'(0)-f'(0)=0$. If $f=l_f+f_0$ where $l_f \in L$ and $f_0 \in D_0$, we have

$$f'(0)=(l_f+f_0)'(0)=l'_f(0)+f'_0(0)=l'_f(0)=l_f.$$

Therefore, this expression is unique.

A linear subset I of a near-algebra $\mathcal{A}$ is called *a left ideal* if $gf \in I$ whenever $f \in I$ and $g \in \mathcal{A}$. A linear subset I of $\mathcal{A}$ is called *a left NA-ideal* if $g(f+h)-gh \in I$ whenever g, $h \in \mathcal{A}$ and $f \in I$.

Theorem 4. *Let $\mathcal{A}$ be a near-algebra of differentiable mappings on a Banach space E. If $\mathcal{A} \supset L$, $f(0)=0$ for every $f \in \mathcal{A}$ and every left ideal is a left NA-ideal, then $\mathcal{A}=L$.*

Proof. Let us consider the following set:

$$D_a=\{f \in \mathcal{A} \mid f'(a)=0\},$$

which is obviously a left ideal of $\mathcal{A}$. Then it follows from our assumption that

$$g(f+h)-gh \in D_a \quad \text{whenever} \quad g,\ h \in \mathcal{A} \quad \text{and} \quad f \in D_a,$$

which means that, putting $h=1$ (the identity mapping),

$$g'(f(a)+a)\,(f'(a)+1)-g'(a)=0.$$

Since $f'(a)=0$, we have

$$g'(f(a)+a)=g'(a) \quad \text{whenever} \quad g \in \mathcal{A} \quad \text{and} \quad f \in D_a.$$

(1) Let us assume that there exist an element $f_0 \in D_a$ and an element $a \in E$ such that $f_0(a) \neq 0$. Let us take $\bar{a} \in \bar{E}$ such that $\bar{a}(f_0(a))=1$. Then, for any element $b \in E$, we have $(b \otimes \bar{a})f_0 \in D_a$, where $b \otimes \bar{a}$ is a linear mapping which is defined by $(b \otimes \bar{a})\,(x)=\bar{a}(x)b$ for every x. Therefore, putting $f=(b \otimes \bar{a})f_0$, since $(b \otimes \bar{a})f_0(a)=b$, we have

$$g'(b+a)=g'(a) \quad \text{for every} \quad b \in E,$$

which means that $g'(x)$ is constant with respect to $x \in E$, or, equivalently, g is a linear mapping. Since g is an arbitrary element of $\mathcal{A}$, we have $\mathcal{A}=L$.

(2) Let us assume that, for any $x \in E$, we have $f(x)=0$ for every $f \in D_x$. Then, since $f-f'(x) \in D_x$, we have

$$f(x)=f'(x)\,(x) \quad \text{for every} \quad x \in E \quad \text{and} \quad f \in \mathcal{A}.$$

Now, let us take an arbitrary $\bar{a} \in \bar{E}$ and consider the functional

$$\Phi(t)=\bar{a}(f(tx)).$$

Then, since

$$\Phi'(t)=\bar{a}(f'(tx)\,(x))=\frac{1}{t}\bar{a}(f(tx))=\frac{1}{t}\Phi(t),$$

we have that $\Phi(t)=ct$ for every real number t and for some constant c. Therefore, we have

$$\bar{a}(f(tx))=t\bar{a}(f(x)),$$

which implies that

$$f(tx)=tf(x) \quad \text{for every} \quad x \in E \quad \text{and number} \quad t,$$

because $\bar{a}$ is an arbitrary element of $\bar{E}$. Now, from Lemma 2 it follows that

$$\begin{aligned} 0 &= f_0'(0)(x) \\ &= \lim_{t\to 0} f_0(tx)/t = \lim_{t\to 0} (f(tx) - l_f(tx))/t \\ &= f(x) - l_f(x) \quad \text{for every} \quad x \in E, \end{aligned}$$

which means that $f = l_f \in L$. Thus the proof is completed.

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